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SIGNAL PROCESSING IN THE LAND OF ART, CULTURE AND BEAUTY

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Sparse Arrays and Sparse Waveforms: Design, Processing, and Applications - Part I -

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Outline

Sparse Arrays and Sparse Waveforms: Design, Processing, and Applications

Part I

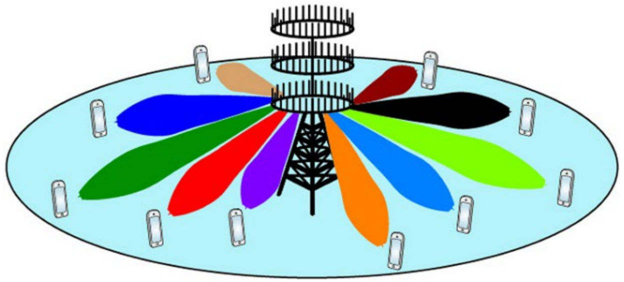
A S P Lab

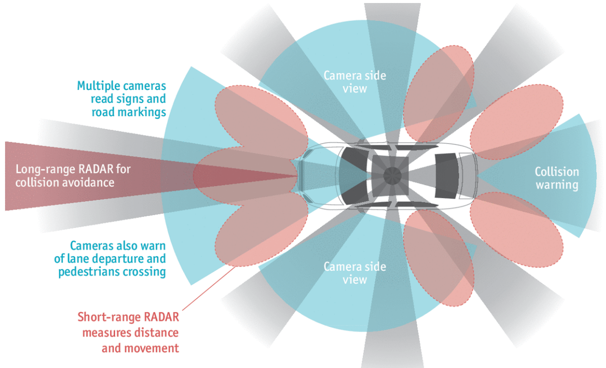
Part I: Introduction

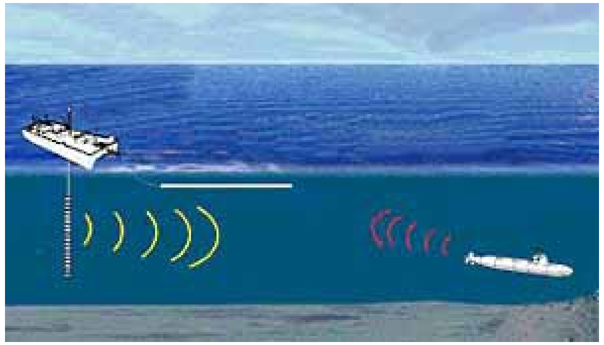
- Direction-of-arrival (DOA) estimation and applications
- Narrowband array signal model
- Beamforming-based DOA estimation

Applications of array processing

- Array processing uses multiple sensors (antennas, microphones, transducers) and plays a fundamental role in wireless communications, radar and sonar sensing, autonomous driving, speech separation, and medical imaging
- **Beamforming**
  - Signal enhancement
  - Interference cancellation
  - Multi-user detection
  - Multiple-input multiple-output (MIMO) systems
  - Increased channel capacity
- **Sensing: Localization/imaging**
  - Ground-to-air radar
  - Automotive radar
  - Sonar
  - Ultrasonic imaging







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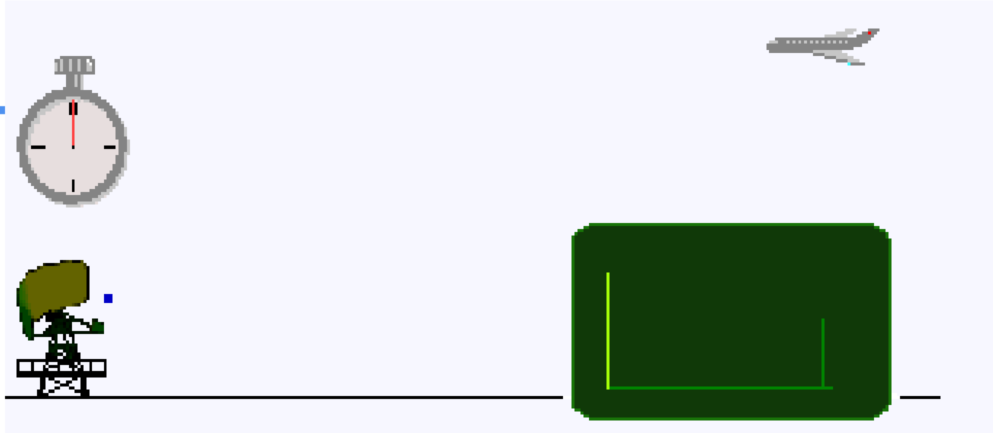
Four-dimensional sensing



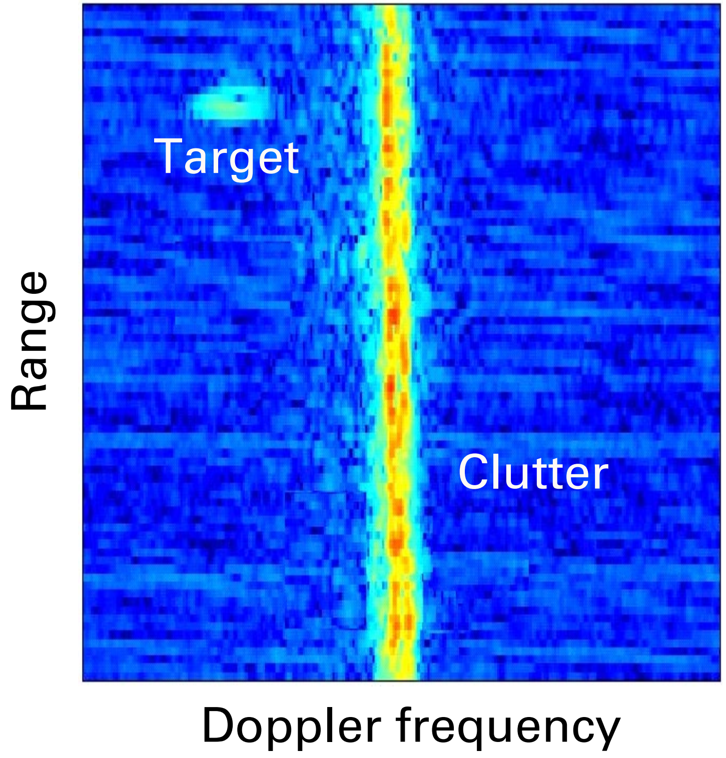
Radar sensing often requires high-resolution results in four dimensions (**4-D imaging**):

- **Range:** range resolution and accuracy are determined by signal bandwidth
- **Doppler frequency:** corresponding to radial velocity with its resolution determined by pulse repetition frequency
- **Azimuth and elevation angles**

We first discuss sparse array design and processing for DOA estimation, and waveform design for range and Doppler estimation will follow:

- more signals
- higher resolution
- few sensors

We mainly consider 1-D DOA estimation using linear arrays. Most results can be easily modified for 2-D DOA estimation with additional complexity.



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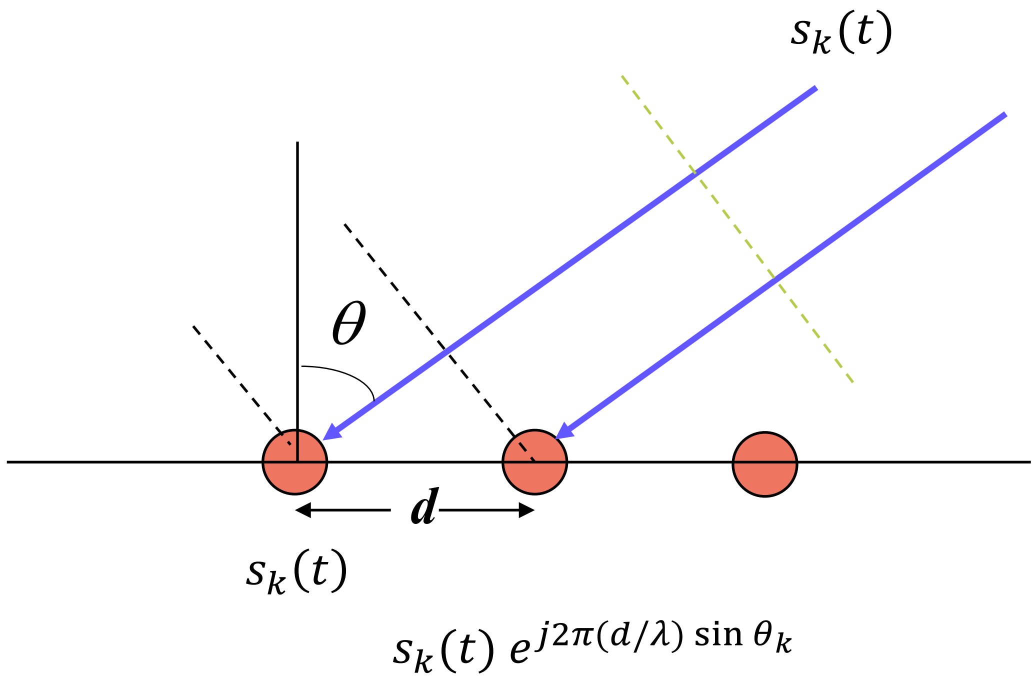
Signal model for ULA

Narrowband array signal model

- uniform linear array (ULA)
- K far-field signals



$$\boldsymbol{x}(t) = \sum_{k=1}^{K} s_k(t)\boldsymbol{a}(\theta_k) + \boldsymbol{n}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t)$$

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ e^{j2\pi(d/\lambda)\sin(\theta)} \\ \vdots \\ e^{j2\pi(N-1)(d/\lambda)\sin(\theta)} \end{bmatrix}$$: steering vector

$\mathbf{A} = [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_K), \cdots, \boldsymbol{a}(\theta_K)]$: array manifold matrix

$\boldsymbol{s}(t) = [s_1(t), \cdots, s_K(t)]^{\mathrm{T}}$: signal vector

DOA estimation: determine the directions of signal arrivals, $\theta_1, \cdots, \theta_K$, from the received signal vector $\boldsymbol{x}(t)$ over (typically) multiple samples $t = 1, \cdots, T$.

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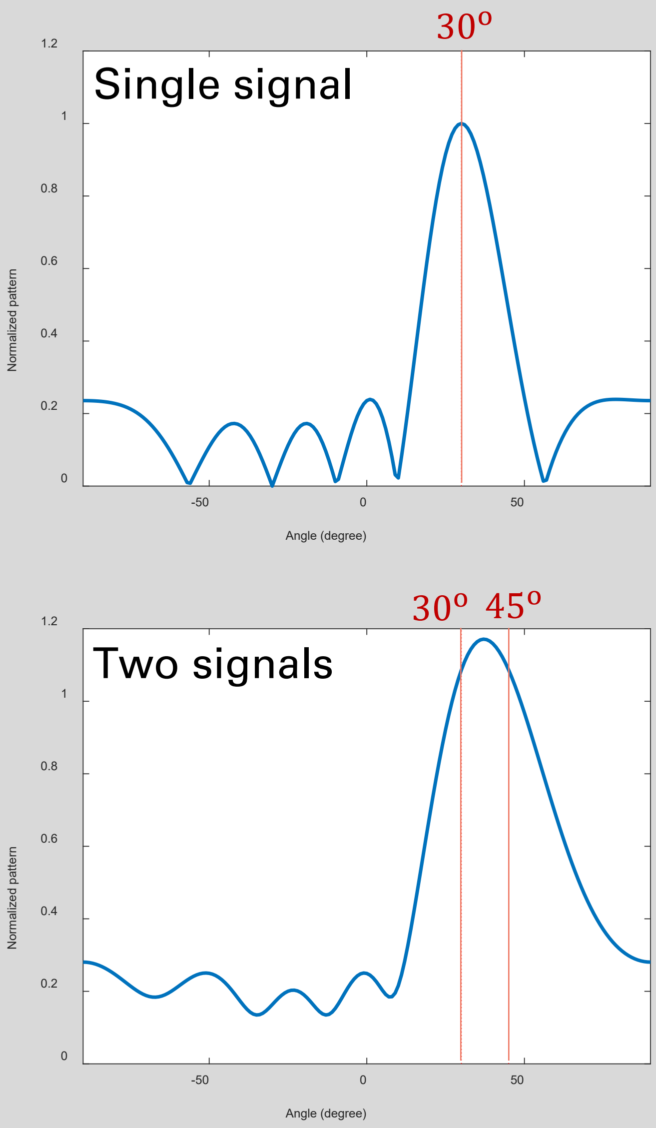
Beamforming-based DOA estimation

Let's first consider **traditional DOA estimation approach through beamforming:**

- For $\mathbf{x}(t) = s(t)\mathbf{a}(\theta_0)$, by assuming $\boldsymbol{w} = \boldsymbol{a}(\theta)$ with different θ, the magnitude of $y(t,\theta) = \boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{x}(t) = s(t)\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{a}(\theta_0)$ is peaked at θ_0.
- This approach has a low resolution because the beamwidth is wide.
- Note that the resolution is determined by the array aperture.

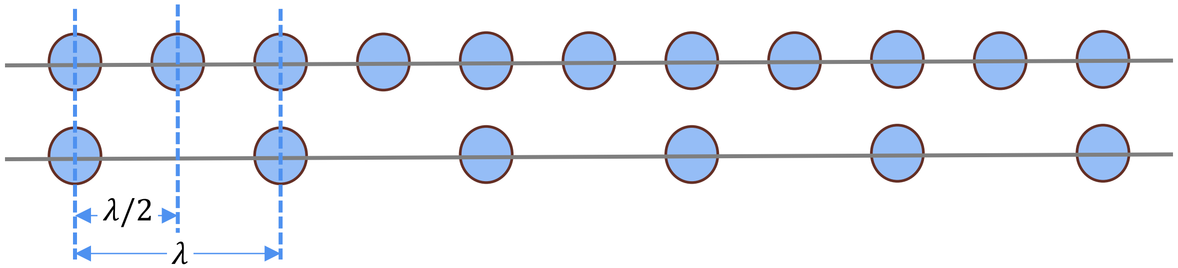
Beamforming-based DOA estimation has a low resolution (Example of 6-element ULA)



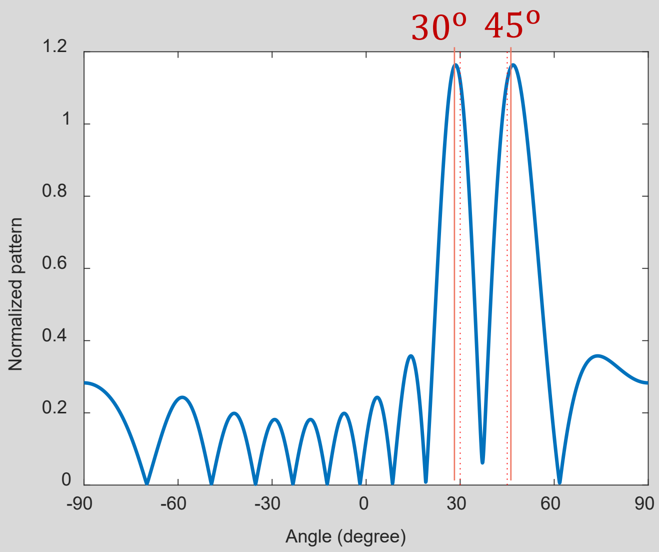
Beamforming-based DOA estimation

Achieving high resolution:

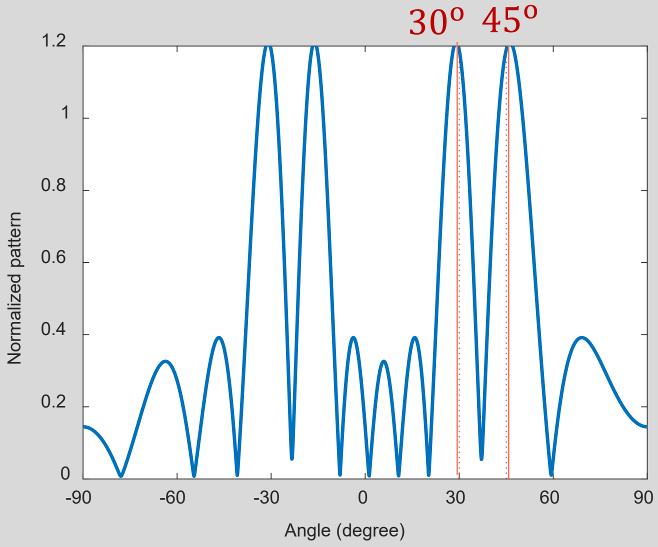
- Larger array aperture
  - More sensors: High cost
  - Large spacing (uniform): Alias
  - **Sparse arrays (irregular)**: To be discussed further



- High-resolution DOA estimation methods
  - Adaptive beamforming (e.g., MVDR)
  - Maximum likelihood estimation
  - Subspace-based methods, e.g., **MUSIC,** ESPRIT
  - **Compressive sensing (sparse reconstruction)**
  - **Machine learning**



11 sensors, $d = \lambda/2$



6 sensors, $d = \lambda$

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Part II: Sparse Array Design and Processing

- More consecutive lags and lower coupling
- Sparsity-based DOA estimation
- Structured matrix completion for DOA estimation

Subspace-based DOA estimation

Subspace-based DOA estimation techniques based on the subspace analysis of the covariance matrix are commonly used to achieve a high resolution.

Eigen-decomposition of the covariance matrix

$$\boldsymbol{R}_{xx} = \boldsymbol{E}[\boldsymbol{x}(t)\boldsymbol{x}^H(t)] = \sum_{i=1}^{K} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^{\mathrm{H}} + \sum_{i=K+1}^{N} \sigma_n^2 \boldsymbol{v}_i \boldsymbol{v}_i^{\mathrm{H}} = \boldsymbol{U}_s \boldsymbol{\Sigma}_s \boldsymbol{U}_s^{\mathrm{H}} + \boldsymbol{U}_n \boldsymbol{\Sigma}_n \boldsymbol{U}_n^{\mathrm{H}}$$

Signal subspace Noise subspace

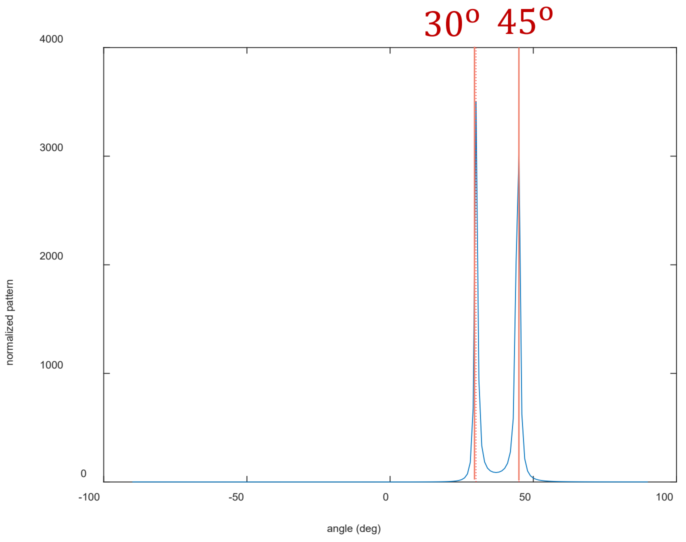
Observations:

- The signal subspace and the noise subspace are orthogonal: $\boldsymbol{U}_s^{\mathrm{H}} \boldsymbol{U}_n = \boldsymbol{0}$.
- Valid signal steering vectors are orthogonal to the noise subspace: $\boldsymbol{A}^{\mathrm{H}} \boldsymbol{U}_n = \boldsymbol{0}$.

Pseudo spatial spectrum of **MUSIC** (MUltiple SIgnal Classification):

$$P(\theta) = \frac{1}{\boldsymbol{a}^{\mathrm{H}}(\theta) \boldsymbol{U}_n \boldsymbol{U}_n^{\mathrm{H}} \boldsymbol{a}(\theta)} = \left\| \boldsymbol{U}_n^{\mathrm{H}} \boldsymbol{a}(\theta) \right\|_2^{-2}$$

- Only 1-D search is needed for multiple signals.
- An N-element ULA can detect $N-1$ signals.
- Knowledge of the number of signals K is required.



R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas and Propagation*, 1986.

Difference coarray

Subspace-based DOA estimation exploits the data covariance matrix $\boldsymbol{R}_{xx}$.

For a ULA with uncorrected signals:

- $\boldsymbol{R}_{xx}$ is Toeplitz (diagonal-constant) and Hermitian
- $\boldsymbol{R}_{xx}$ is highly redundant: Only N elements are unique in the $N \times N$ covariance matrix
- We may not need N sensors to estimate the $N \times N$ covariance matrix

Consider removing the third sensor from a 4-element ULA:

- All the entries of the ULA covariance matrix can be recovered: e.g., $\mathrm{E}[x_2 x_3^*] \Rightarrow \mathrm{E}[x_1 x_2^*]$.
- The 4-element ULA and the 3-element sparse array are **different coarray equivalent** because they generate the same number of correlation lags.
- For physical array $\mathbb{G}$, the difference lags are given as: $\mathbb{C}_G = \{\boldsymbol{z} | \boldsymbol{z} = \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{G}\}$.

$$\boldsymbol{R}_{xx} = \begin{bmatrix} \mathrm{E}[x_1 x_1^*] & \mathrm{E}[x_1 x_2^*] & \mathrm{E}[x_1 x_3^*] & \mathrm{E}[x_1 x_4^*] \\ \mathrm{E}[x_2 x_1^*] & \mathrm{E}[x_2 x_2^*] & \mathrm{E}[x_2 x_3^*] & \mathrm{E}[x_2 x_4^*] \\ \mathrm{E}[x_3 x_1^*] & \mathrm{E}[x_3 x_2^*] & \mathrm{E}[x_3 x_3^*] & \mathrm{E}[x_3 x_4^*] \\ \mathrm{E}[x_4 x_1^*] & \mathrm{E}[x_4 x_2^*] & \mathrm{E}[x_4 x_3^*] & \mathrm{E}[x_4 x_4^*] \end{bmatrix}$$

Correlation lags (difference coarray)

Direct MUSIC-based DOA estimation

Vectorizing $\boldsymbol{R}_{xx}$ yields

$$\boldsymbol{z} = \text{vec}(\boldsymbol{R}_{xx}) = \widetilde{\boldsymbol{A}}\,\boldsymbol{b} + \sigma_n^2\,\tilde{\boldsymbol{\imath}} = \boldsymbol{A}^o\boldsymbol{b}^o$$

$\widetilde{\boldsymbol{A}} = \left[\boldsymbol{a}(\theta_1)\otimes\boldsymbol{a}^*(\theta_1), \ldots, \boldsymbol{a}(\theta_Q)\otimes\boldsymbol{a}^*(\theta_Q)\right]$: Manifold matrix for the difference coarray

$\boldsymbol{b} = \left[\sigma_1^2, \ldots, \sigma_Q^2\right]^{\mathrm{T}}$: Source power vector

$\tilde{\boldsymbol{\imath}} = \text{vec}(\boldsymbol{I}_N)$

$\boldsymbol{A}^o = [\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{I}}]$

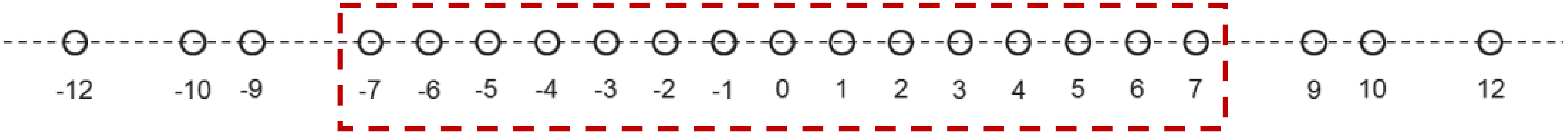
$\boldsymbol{b}^o = [\boldsymbol{b}^{\mathrm{T}}, \sigma_{\mathrm{n}}^2]^{\mathrm{T}}$

$\boldsymbol{z}$ acts as received data of a virtual array (difference coarray)

- Manifold matrix corresponds to virtual sensors which are much more than physical antennas
- Only a single snapshot corresponding to vector $\boldsymbol{b}$
- Subspace-based DOA estimation cannot be directly applied because the single-snapshot covariance matrix $\boldsymbol{z}\boldsymbol{z}^H$ is rank-1

P. Pal and P. P. Vaidyanathan, “Coprime sampling and the MUSIC algorithm,” *IEEE Digit. Signal Process. Workshop/ IEEE Signal Process. Educ. Workshop*, 2011.

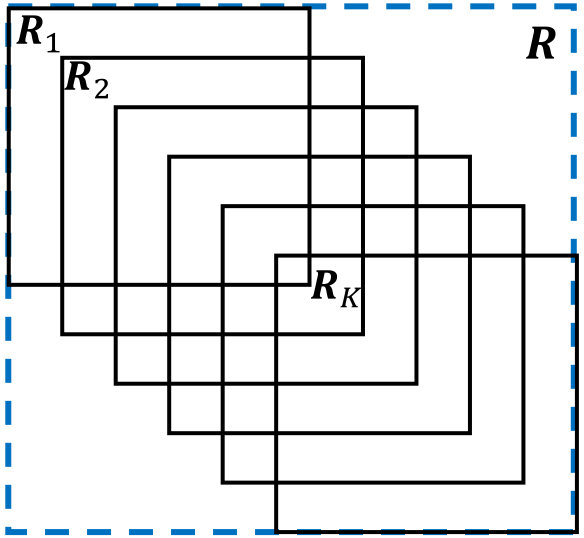
Direct MUSIC-based DOA estimation



Spatial smoothing

Example: $N_z = 15, K = 8$

- By dividing the rank-1 matrix $\boldsymbol{R}$ into $K = (N_z + 1)/2$ subarrays $\boldsymbol{R}_k$, their average $\frac{1}{K}\sum_{k=1}^{K} \boldsymbol{R}_k$ becomes rank-K.
- It is equivalent to placing the elements of $\boldsymbol{z} = \left[z_{-(K-1)}, \cdots, z_{K-1}\right]^T$ in a Hermitian and Toeplitz manner.
- Only consecutive lags can be used for this purpose (e.g., lags of [−7:1:7]; detect up to 7 signals).
- In this context, **optimum design** of parse arrays is to
  - A high number of consecutive lags
  - Low mutual coupling (few lag-1 and lag-2 pairs)



$$\widetilde{\boldsymbol{R}}_{xx} = \begin{bmatrix} z_0 & z_1 & \cdots & z_{K-1} \\ z_{-1} & z_0 & \cdots & z_{K-2} \\ \vdots & \vdots & \ddots & \vdots \\ z_{-(K-1)} & z_{-(K-2)} & \cdots & z_0 \end{bmatrix}$$

P. Pal and P. P. Vaidyanathan, "Coprime sampling and the MUSIC algorithm," *IEEE Digit. Signal Process. Workshop/ IEEE Signal Process. Educ. Workshop*, 2011.

C.-L. Liu and P. P. Vaidyanathan, "Remarks on the Spatial Smoothing Step in Coarray MUSIC," *IEEE Signal Processing Letters*, 2015.

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Minimum redundancy array

Minimum redundancy array (MRA): For a given number of physical sensors, MRA maximizes the number of consecutive virtual sensors in the resulting difference coarray.

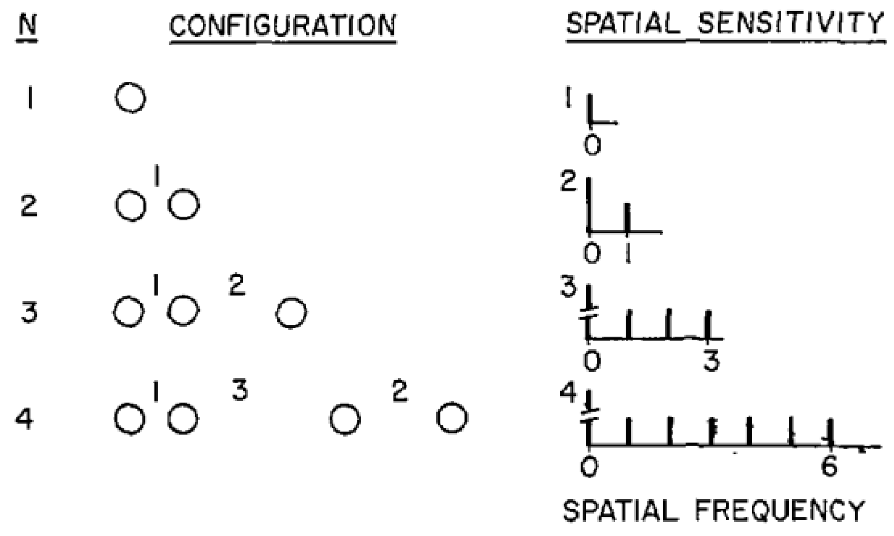
- **Restricted arrays:** All lags are consecutive
- **General arrays:** Not all lags are consecutive

The difference lags an N-element sparse array can achieve is in the order of $\frac{1}{2}N(N-1)$.

For an MRA, the redundancy is defined as $R = \frac{\frac{1}{2}N(N-1)}{N_{\max}}$, where $N_{\max}$ is the maximum number of consecutive lags.

- R is found to be $1.217 \lesssim R \lesssim 1.674$.

However, MRA cannot be systematically designed.



N	N_{max}	R	Configuration
Restricted Arrays:			
5	9	1.11	·1·3·3·2·
6	13	1.16	·1·5·3·2·2
7	17	1.24	·1·3·6·2·3·2·
8	23	1.22	·1·3·6·6·2·3·2·
9	29	1.24	·1·3·6·6·6·2·3·2·
10	36	1.25	·1·2·3·7·7·7·4·4·1·
General Arrays:			
5	9	1.11	·4·1·2·6·
6	13	1.16	·6·1·2·2·8·
7	18	1.17	·14·1·3·6·2·5·
8	24	1.17	·8·10·1·3·2·7·8·
10	37	1.22	·16·1·11·8·6·4·3·2·22·

A. Moffet, "Minimum-redundancy linear arrays," *IEEE Trans. Antennas and Propagation*, 1968.
M. Ishiguro, "Minimum redundancy linear arrays for a large number of antennas," *Radio Science*, 1980.

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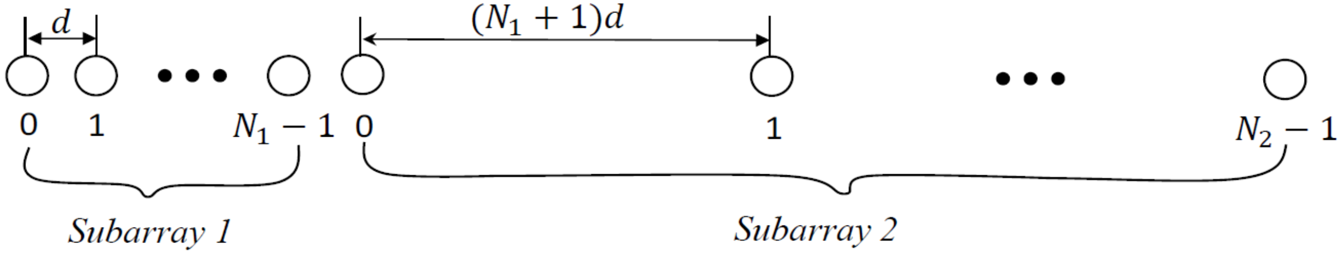
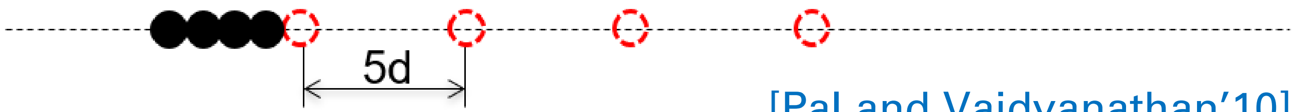
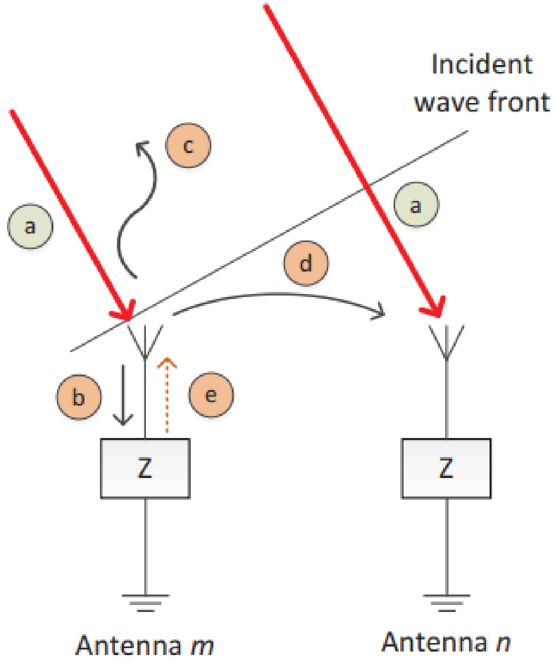
Systematical sparse array design: Nested array

Systematical design: **Nested array** is a simple sparse array configuration which consists of two uniform linear subarrays, one of which has a unit spacing.





[Pal and Vaidyanathan'10]

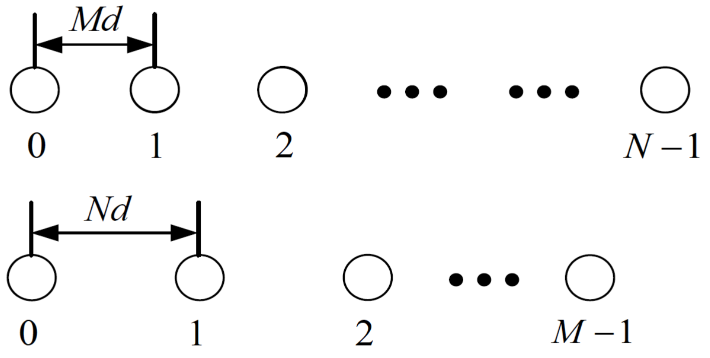


- For nested arrays, all lags are consecutive.
- Depending on the applications, the high number of consecutive physical sensors may cause high **mutual coupling effect**, degrading DOA estimation performance.
- Mutual coupling brings higher impact when the interelement spacing is small (e.g., half-wavelength spacing or less).
- The coprime array is proposed as an alternative to nested array.

P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Processing*, 2010.

Systematical sparse array design: Coprime array

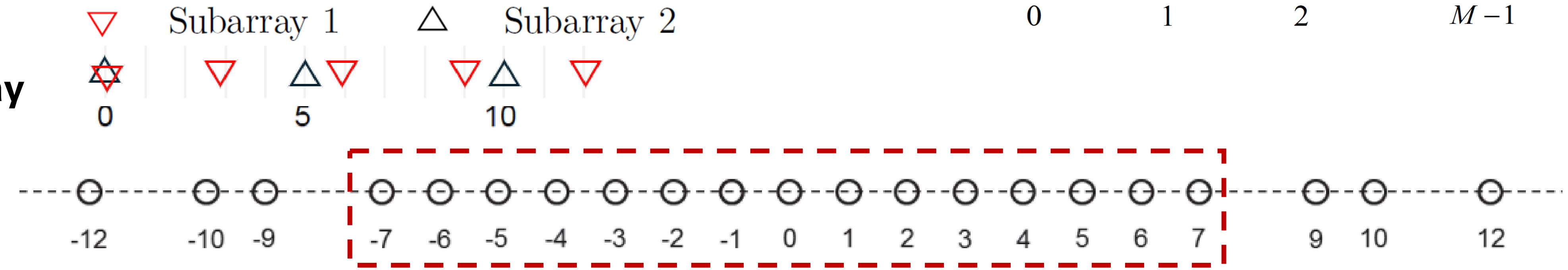
Coprime array: utilizes a pair of uniform linear subarrays with M and N being coprime integers (greatest common divisor $\gcd(M,N) = 1$).



Example: $M = 3$ and $N = 5$ (6 elements)

Physical array

Coarray



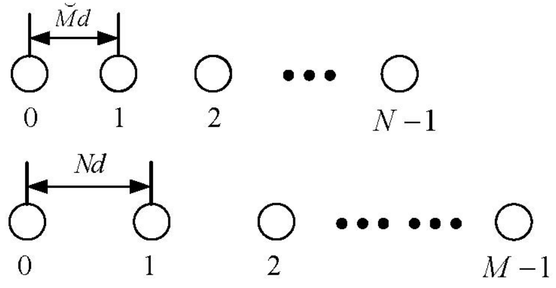
- Unlike nested arrays, coprime arrays generally have holes in the resulting lags.

P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime sampler and arrays," *IEEE Trans. Signal Processing*, 2011

Generalized coprime arrays: CACIS

Problems:

- Coprime array: have holes in the lags
- Nested array: high mutual coupling



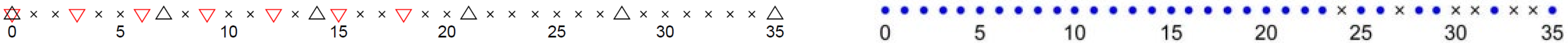
CACIS (Coprime array with compressed inter-element spacing):

Compresses the interelement spacing of one subarray $\breve{M} = M/p$ with $2 \leq p \leq M$ to increase the number of consecutive lags η_c

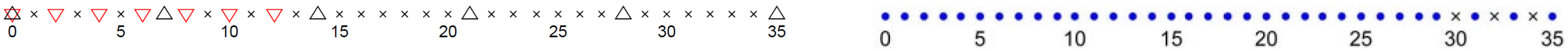
Example: $M = 6, N = 7$

(Only showing nonnegative lags)

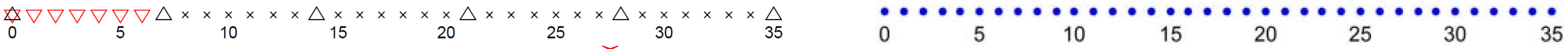
$\breve{M} = 6, p = 1, N = 7, \eta_c = 25$ (prototype)



$\breve{M} = 3, p = 2, N = 7, \eta_c = 47$



$\breve{M} = 2, p = 3, N = 7, \eta_c = 59$



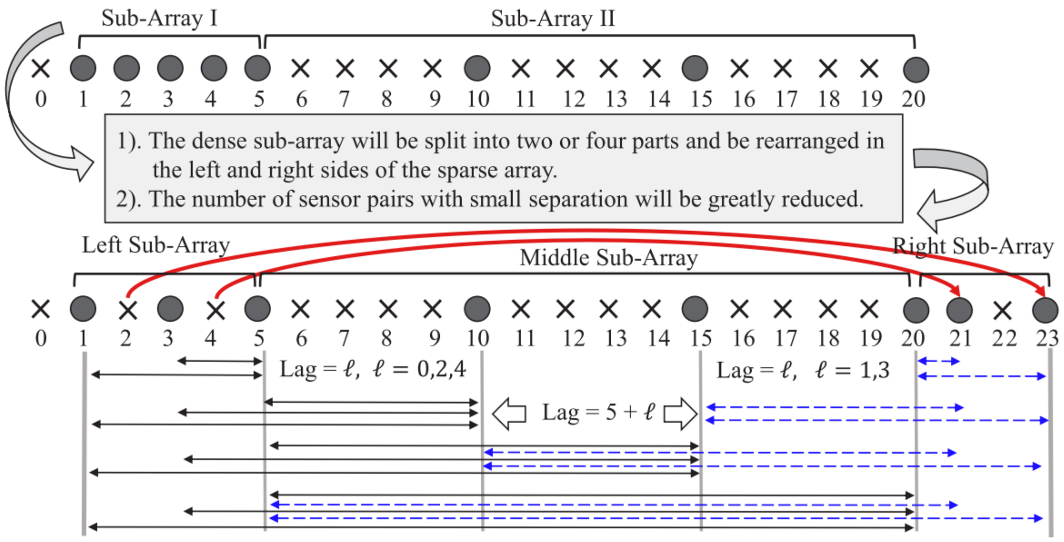
$\breve{M} = 1, p = 6, N = 7, \eta_c = 71$ (nested)

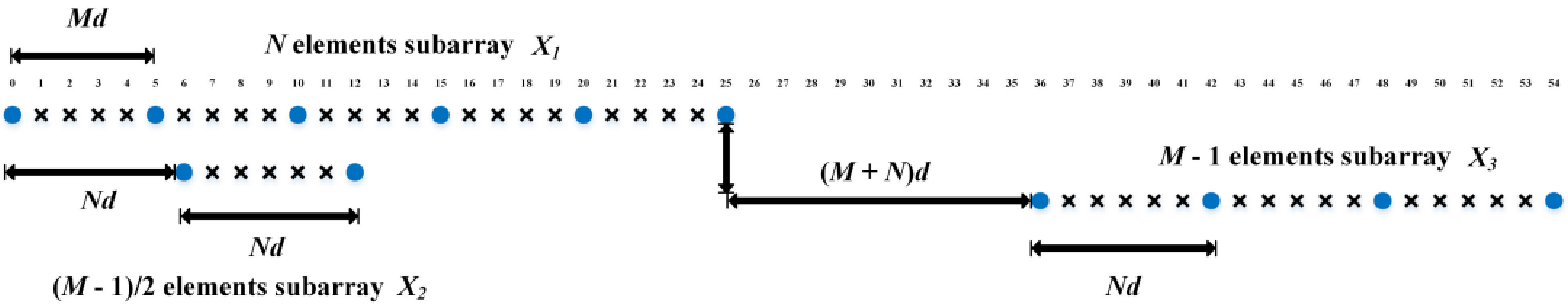
S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Processing*, 2015.

More consecutive lags and less mutual coupling

Many sparse arrays are proposed for (i) more consecutive lags and (ii) less mutual coupling.

- **Augmented nested array:** Split the densely located elements in inner subarray to reduce the mutual coupling. Several variations.
- **Thinned Coprime Array:** Provides the same number of consecutive lags, unique lags, and aperture as the conventional coprime array but with fewer sensors.





J. Liu, Y. Zhang, Y. Lu, S. Ren and S. Cao, "Augmented nested arrays with enhanced DOF and reduced mutual coupling," *IEEE Trans. Signal Processing*, 2017.

A. Raza, W. Liu and Q. Shen, "Thinned coprime array for second-order difference co-array generation with reduced mutual coupling," *IEEE Trans. Signal Processing*, 2019.

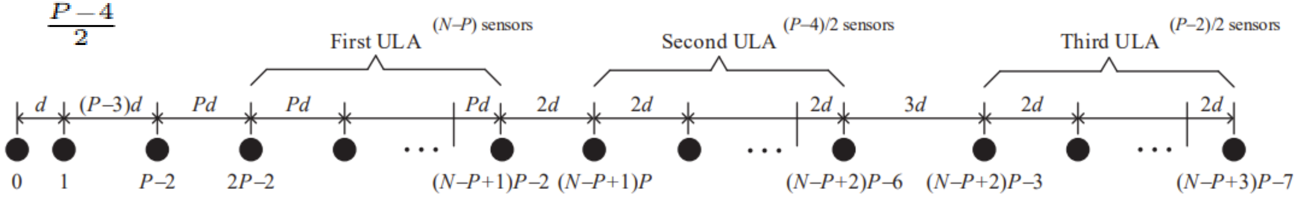
Sparse array: MISC family

MISC (maximum interelement spacing constraint): A four-segment configuration to achieve a high number of consecutive lags with low mutual coupling

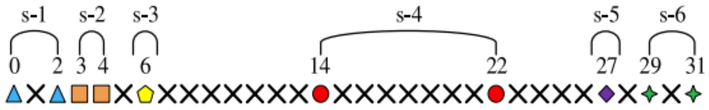
$$\mathbb{A}_{\text{MISC}} = \{1, P-3, \underbrace{P,\dots,P}_{N-P}, \underbrace{2,\dots,2}_{\frac{P-4}{2}}, 3, \underbrace{2,\dots,2}_{\frac{P-4}{2}}\}$$ with $P = 2\lfloor N/4 \rfloor + 2$ $(N \geq 5)$

Modified versions: Use more segments to achieve higher freedom and better performance

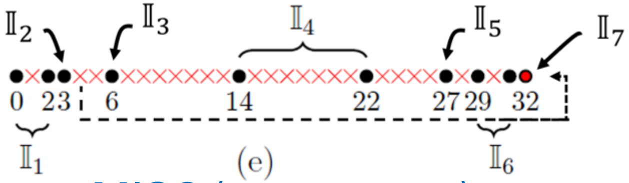
- Improved MISC (I-MISC)
- Enhanced MISC (EMISC)
- Symmetry improved MISC (S-IMISC)
- Extended MISC (xMISC)



MISC (4 segments)



I-MISC (6 segments)



xMISC (7 segments)

Z. Zheng, W-Q. Wang, Y. Kong, and Y. D. Zhang, "MISC Array: A new sparse array design achieving increased degrees of freedom and reduced mutual coupling effect," *IEEE Trans. Signal Processing*, 2019.

W. Shi, Y. Li, and R. C. de Lamare, "Novel sparse array design based on the maximum inter-element spacing criterion," *IEEE Signal Processing Letters*, 2022.

X. Sheng, D. Lu, Y. Li, and R. C. de Lamare, "Enhanced MISC-based sparse array with high uDOFs and low mutual coupling," *IEEE Trans. Circuits and Systems II: Express Briefs*, 2024.

X. Li, H. Yang, J. Han, and N. Dong, "A novel low-complexity method for near-field sources based on an S-IMISC array model," *Electronics*, 2023.

S. Wandale and K. Ichige, "xMISC: Improved sparse linear array via maximum inter-element spacing concept," *IEEE Signal Processing Letters*, 2023.

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Sparse array: Performance evaluation

DOF ratio:

$$\gamma(N) = \frac{N^2}{\mathcal{S}_u}$$

$\mathcal{S}_u$: one-side uniform DOF (uDOF)

Coupling leakage:

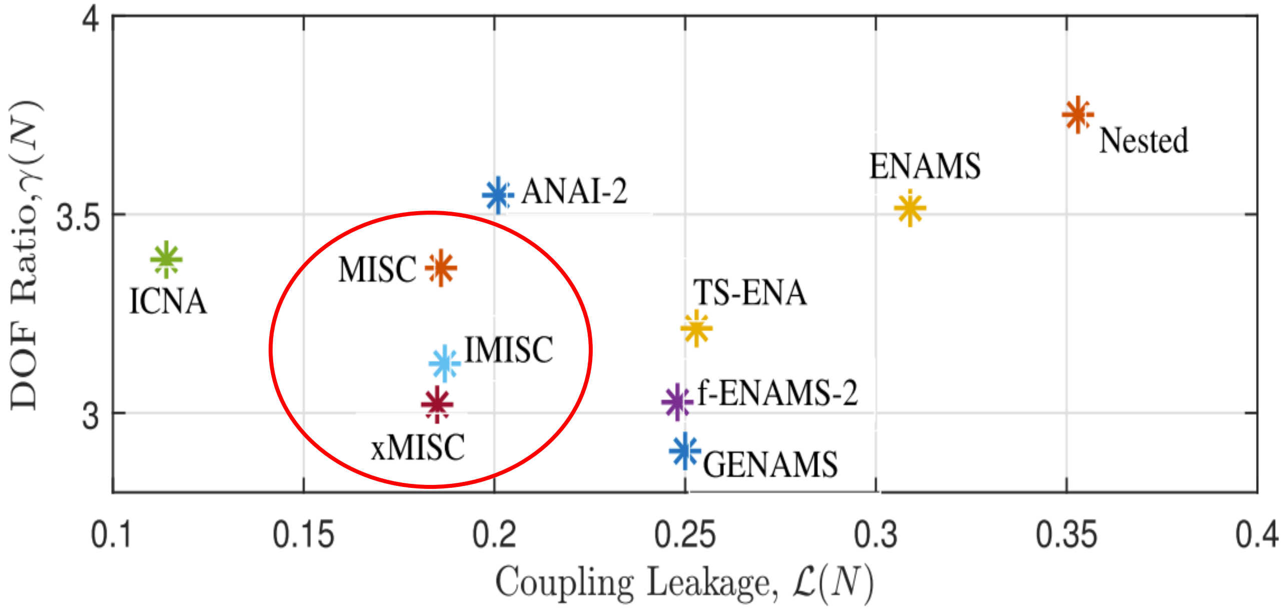
$$\mathcal{L}(N) = \frac{\|\boldsymbol{H} - \mathrm{diag}(\boldsymbol{H})\|_\mathrm{F}}{\|\boldsymbol{H}\|_\mathrm{F}}$$

$\boldsymbol{H}$: mutual coupling matrix whose elements depends on the distance between elements

Simulations assumed

$$\langle \boldsymbol{H} \rangle_{j,l} = \begin{cases} c_{|j-l|}, & \text{if } |j-k| \leq V \\ 0, & \text{otherwise} \end{cases}$$

with $c_0 = 1$, $c_1 = 0.2e^{j\pi/3}$, $\left|\frac{c_b}{c_l}\right| = \frac{l}{b}$



(N=30)

S. Wandale and K. Ichige, “xMISC: Improved sparse linear array via maximum inter-element spacing concept,” *IEEE Signal Processing Letters*, 2023.

Sparse signal reconstruction

- Compressive sensing problems are expressed as

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_0 \quad \text{s.t. } \boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x}$$

- Considering noise, a more general expression is

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_0 \quad \text{s.t.} \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2 \leq \epsilon$$

  where $\epsilon > 0$ specifies the tolerable bound.

- Because of the l_0 norm operation, such problems are non-convex and NP-hard.

- **Greedy algorithms**

  Greedy construction of "support" (=column combination) by adding one-by-one/best choice at each iteration: Orthogonal matching pursuit (OMP), …

- **Convex relaxation**

  Approximation of the cost by convex functions (typically l_1-norm recovery): LASSO (least absolute shrinkage and selection operator), …

- **Probabilistic inference**

  (Approximate) employment of probabilistic inference: Bayesian compressive sensing (sparse Bayesian learning)

Sparsity-based DOA estimation

Sparsity-based DOA estimation:

$$\boldsymbol{z} = \mathrm{vec}(\boldsymbol{R}_{xx}) = \widetilde{\boldsymbol{A}}\,\boldsymbol{b} + \sigma_n^2 \tilde{\boldsymbol{i}} = \boldsymbol{A}^o \boldsymbol{b}^o$$

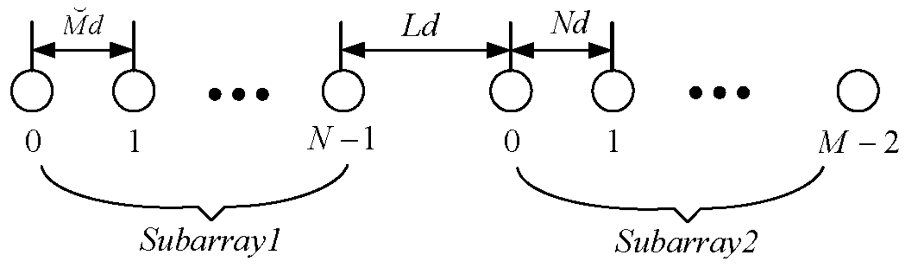
- The linear coarray model well fits into the compressive sensing problem by defining dense dictionary matrix $\boldsymbol{A}^g$ over a grid, e.g., $[-90{:}\,1{:}\,90]$:

$$\min_{\boldsymbol{z}} \|\boldsymbol{b}^g\|_0 \ \text{s.t.}\ \|\boldsymbol{z} - \boldsymbol{A}^g \boldsymbol{b}^g\|_2 \le \epsilon$$

- The positions of the nonzero solutions of $\boldsymbol{b}^g$ represent the signal DOA
- This approach does not require a specific array structure (e.g., consecutive coarray lags) and all difference lags can be utilized in sparsity-based DOA estimation: **Unique lags**

CADiS (Coprime array with displaced subarrays):

- Displaces two subarrays to increases unique lags
- Very low mutual coupling
- In general, the resulting lags are disconnected in the center region



$\breve{M} = M/p$ with $2 \le p \le M$

S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Processing*, 2015.

Sparse array: CADiS

In CADiS configurations, the self-lags are less likely to coincide with the cross-lags:

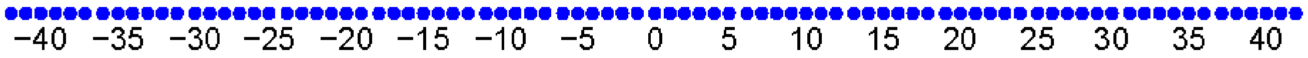
(a) $L > (M-2)N$ achieves the maximum number of unique lags

(b) $L = \breve{M} + N$ yields the largest number of consecutive lags

η_c: consecutive lags; η_u: unique lags

$\breve{M} = 3, p = 2, N = 7, L = \breve{M} + N, \eta_c = 33, \eta_u = 89$

$\breve{M} = 2, p = 3, N = 7, L = \breve{M} + N, \eta_c = 38, \eta_u = 87$



$\breve{M} = 1, p = 6, N = 7, L = \breve{M} + N, \eta_c = 85, \eta_u = 85$

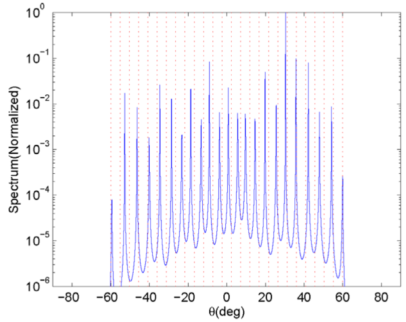
- A smaller value of $\breve{M}$ reduces the unique lags and reduces the number of holes
- The lags become consecutive when $\breve{M} = 1$ (nested array)

S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Processing*, 2015.

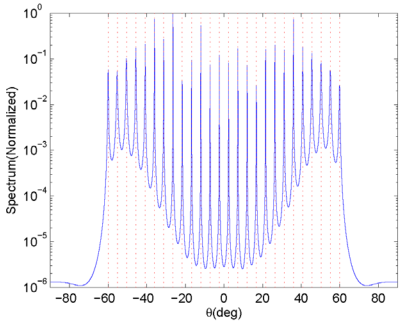
Sparse arrays: Comparison

Consider $M = 6$ and $N = 7$ with $M + N - 1 = 12$ physical sensors

- LASSO-based method achieves better DOA estimation performance
- When LASSO is used, CADiS generally outperforms CACIS

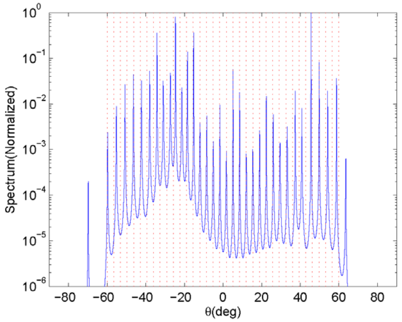


(a) CACIS with $\breve{M} = 3$ ($\eta_c = 47$)

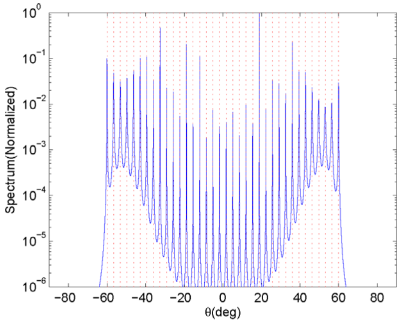


(b) CACIS with $\breve{M} = 2$ ($\eta_c = 59$)

MUSIC (26 signals)

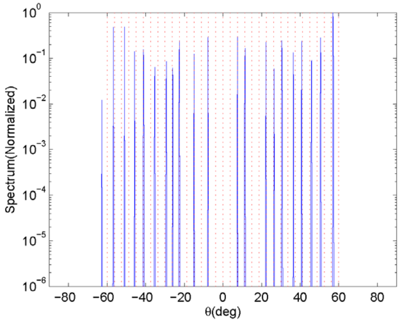


(a) CACIS with $\breve{M} =1$ ($\eta_c = 71$)

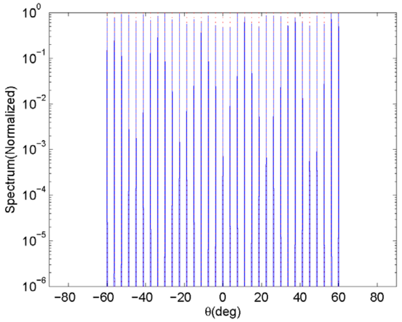


(b) CADiS with $\breve{M} = 1$ ($\eta_c = 85$)

MUSIC (36 signals)



(a) CACIS with $\breve{M} = 2$ ($\eta_u = 65$)



(b) CADiS with $\breve{M} = 2$ ($\eta_u = 87$)

LASSO (33 signals)

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Off-grid problem

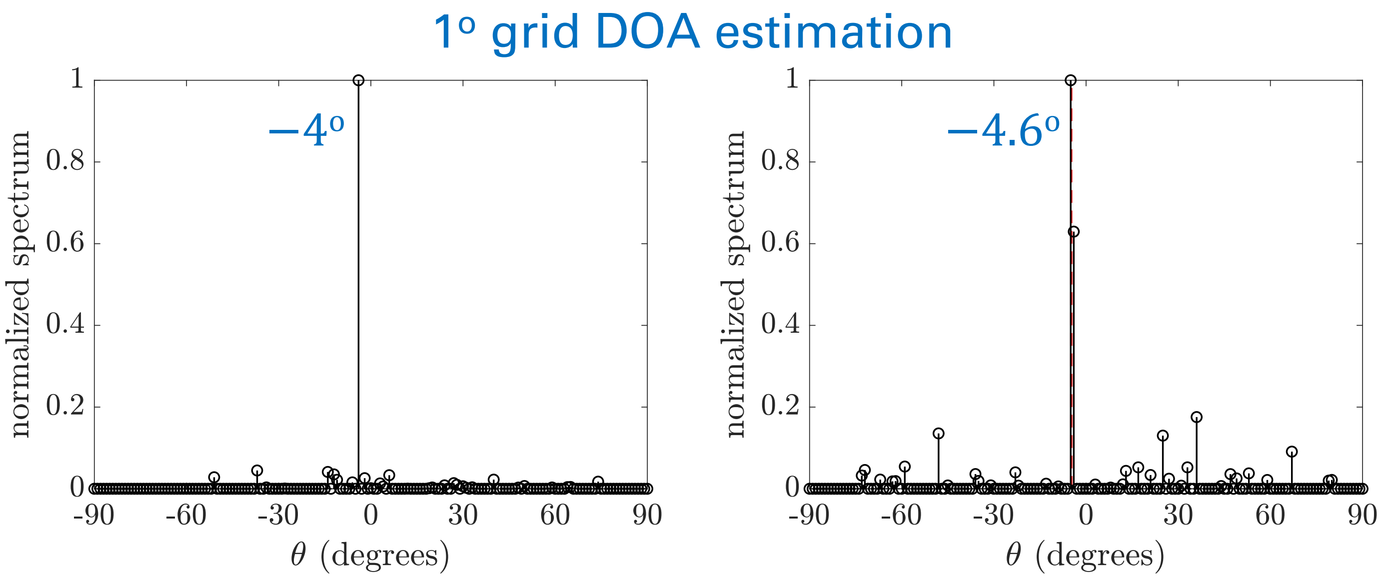
A major problem with the compressive sensing-based DOA estimation approach is that the DOAs must be on the defined grid, e.g., $[-90^\circ : 1^\circ : 90^\circ]$.

Signals arriving from other DOAs will suffer the **off-grid problem**, e.g., signal from -4.6°.

- Less sparse solution
- Difficult to converge

Solutions in the context of compressive sensing:

- Finer grid resolution
- Grid refining
- Off-grid estimation



An attractive method is to **complete the covariance matrix** (matrix completion) so that conventional subspace-based methods (e.g., MUSIC) can be applied.

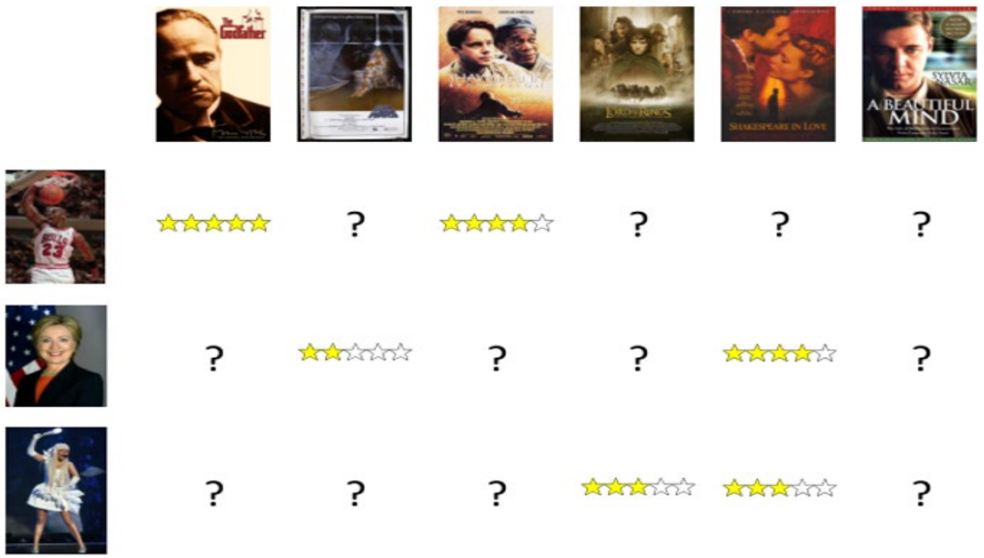
Matrix completion

Netflix problem: Predict unknown scores.

The data is **low-rank**, but the dictionary is unknown (unlike CS).

Let Ω be the region where the elements of matrix $\boldsymbol{M}$ are observed (i.e., $\{M_{ij} | (i,j) \in \Omega\}$), matrix completion finds a low-rank full matrix $\boldsymbol{X}$ which matches $\boldsymbol{M}$:

$$\min_{\boldsymbol{X}} \quad \mathrm{rank}(\boldsymbol{X}) \quad \text{subject to} \quad X_{ij} = M_{ij} \quad \forall\, (i,j) \in \Omega$$



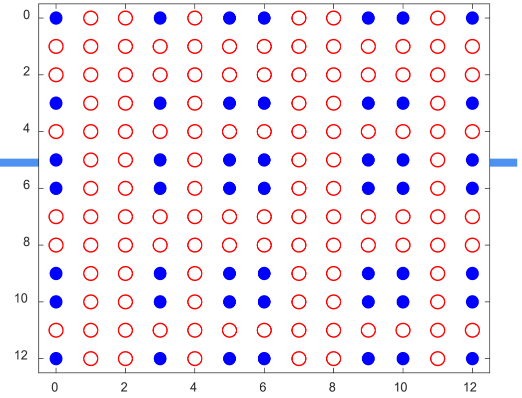
Because the problem involves matrix rank, it is non-convex and NP-hard.

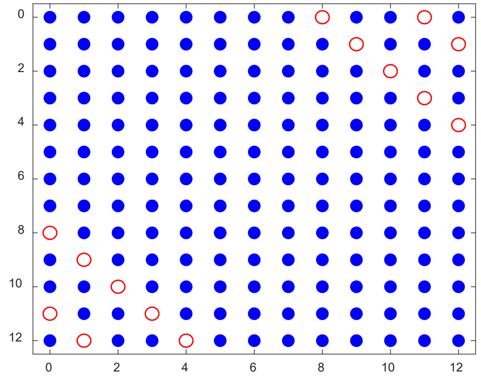
Therefore, the matrix rank is often relaxed, e.g., to its **nuclear norm**:

$$\min_{\boldsymbol{X}} \quad ||\boldsymbol{X}||_* \quad \text{s.t.} \quad X_{ij} = M_{ij} \quad \forall\, (i,j) \in \Omega$$

where $||\boldsymbol{X}||_* = \mathrm{tr}(\sqrt{\boldsymbol{X}^H \boldsymbol{X}})$: nuclear norm of matrix $\boldsymbol{X}$

Structured matrix completion of covariance matrix





A matrix cannot be completed when an entire row or column is missing in the observed matrix.

- Cannot complete covariance matrix of physical array
- However, for ULA, we can recover the covariance matrix utilizing its Toeplitz and Hermitian structure
- The completed covariance matrix can be defined by only a single column vector $\boldsymbol{w}$ as $\mathcal{T}(\boldsymbol{w})$, and obtained from the nuclear norm minimization

$$\min_{\mathbf{z}} \quad \left\| \left[\mathcal{T}(\boldsymbol{w}) - \boldsymbol{R}_{\boldsymbol{y}} \right] \circ \boldsymbol{B}_{\Omega} \right\|_F^2 + \xi ||\mathcal{T}(\boldsymbol{w})||_*$$
$$\text{subject to} \quad \mathcal{T}(\boldsymbol{w}) \succcurlyeq \mathbf{0}$$

where

$\boldsymbol{B}_{\Omega}$: mask matrix with $[\boldsymbol{B}_{\Omega}]_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \Omega \\ 0, & \text{otherwise} \end{cases}$

$\boldsymbol{R}_{\boldsymbol{y}}$: observed sparse covariance matrix (nonzero only in $\boldsymbol{B}_{\Omega}$)

ξ: regularization parameter

In a Hermitian Toeplitz matrix, a single column w uniquely specifies all the elements of the matrix.

$$\mathcal{T}(\boldsymbol{w}) = \begin{bmatrix} w_1 & w_2^* & w_3^* & w_4^* \\ w_2 & w_1 & w_2^* & w_3^* \\ w_3 & w_2 & w_1 & w_2^* \\ w_4 & w_3 & w_2 & w_1 \end{bmatrix}$$

$\boldsymbol{w}$

H. Qiao and P. Pal, "Unified analysis of co-array interpolation for direction-of-arrival estimation," IEEE ICASSP, 2017.

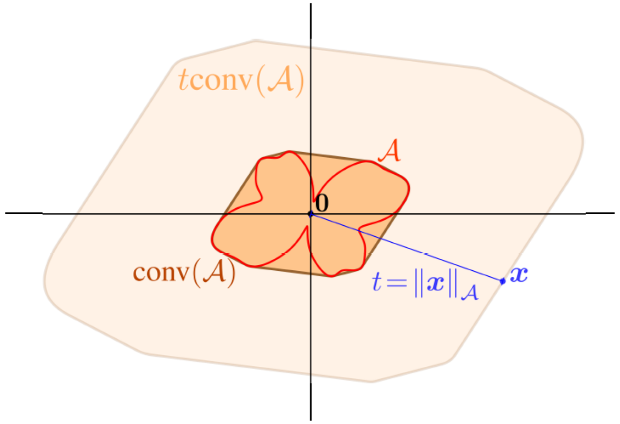
C. Zhou, Y. Gu, Z. Shi, and Y. D. Zhang, "Off-grid direction-of-arrival estimation using coprime array interpolation," *IEEE Signal Processing Letters*, 2018.

Structured matrix completion of covariance matrix

Atomic norm

- A more general alternative is based on the minimization of the atomic norm.
- Using **atomic set** $\mathcal{A} = \{\boldsymbol{a}_i\}$, an observation vector can be expressed as

$$\boldsymbol{x} = \sum_i c_i \boldsymbol{a}_i, \qquad \boldsymbol{a}_i \in \mathcal{A}$$



- The atomic norm of $\boldsymbol{x}$ is defined as

$$\|\boldsymbol{x}\|_{\mathcal{A}} = \inf\{t \geq 0 : x \in t \cdot \mathrm{conv}(\mathcal{A})\}$$

where $\mathrm{conv}(\mathcal{A})$ is the convex hull of $\mathrm{conv}(\boldsymbol{\mathcal{A}})\}$

Rank minimization-based structure matrix reconstruction

- Both nuclear and atomic norm minimization problems approximate rank minimization.
- To prevent the approximation loss, the rank function can be reformulated a multi-convex form.

C. Zhou, Y. Gu, X. Fan, Z. Shi, G. Mao, and Y. D. Zhang, "Direction-of-arrival estimation for coprime array via virtual array interpolation," *IEEE Trans. Signal Processing*, 2018.
Y. Chi and M. Ferreira Da Costa, "Harnessing sparsity over the continuum: Atomic norm minimization for superresolution," *IEEE Signal Processing Magazine*, 2020.
S. Liu, Z. Mao, Y. D. Zhang, and Y. Huang, "Rank minimization-based Toeplitz reconstruction for DoA estimation using coprime array," *IEEE Communications Letters*, July 2021.

ASP Lab

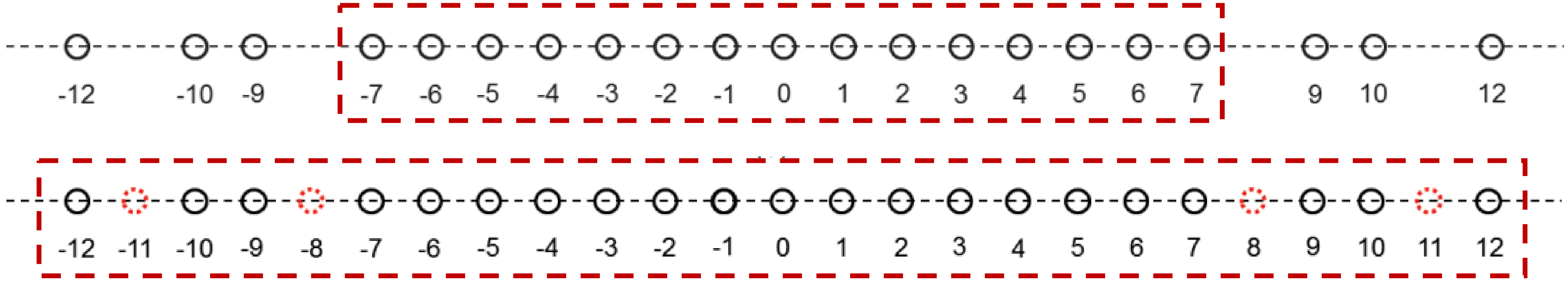
Matrix completion-aware sparse array design

Matrix completion

- Fills in information in missing lags
- Converts missing holes in the lag from obstacles in consecutive-lag construction into a resource for aperture extension
- Enabling off-grid DOA estimation with larger array apertures

Direct MUSIC

MUSIC with matrix completion



With such capability, how shall we consider the "optimality" of a sparse array?

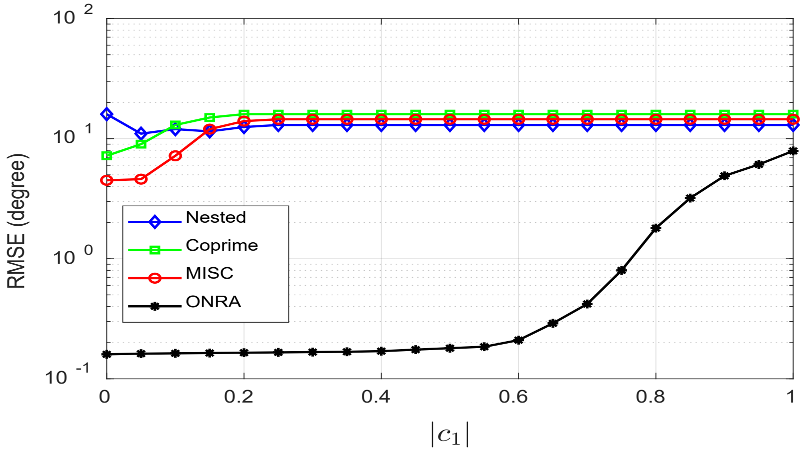
We introduce **optimized non-redundant array (ONRA)**:

- Redundancy-free: Each lag only appears once (except lag-0)
- Introduce holes in the lag for reducing mutual coupling and enlarging array aperture
- Optimized using mixed-integer linear programming approach (not systematical)

A. Ahmed and Y. D. Zhang, "Generalized non-redundant sparse array designs," *IEEE Trans. Signal Processing*, 2021.

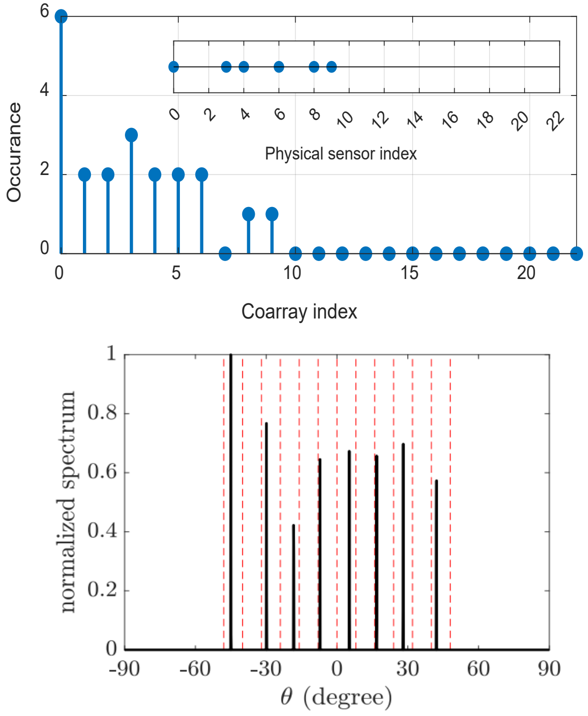
Non-redundant sparse array: Comparison

- Comparison for 6-sensor arrays (DOA estimation for 13 sources; LASSO)
- ONRA has very low mutual coupling effect as the minimum interelement spacing is 2 units

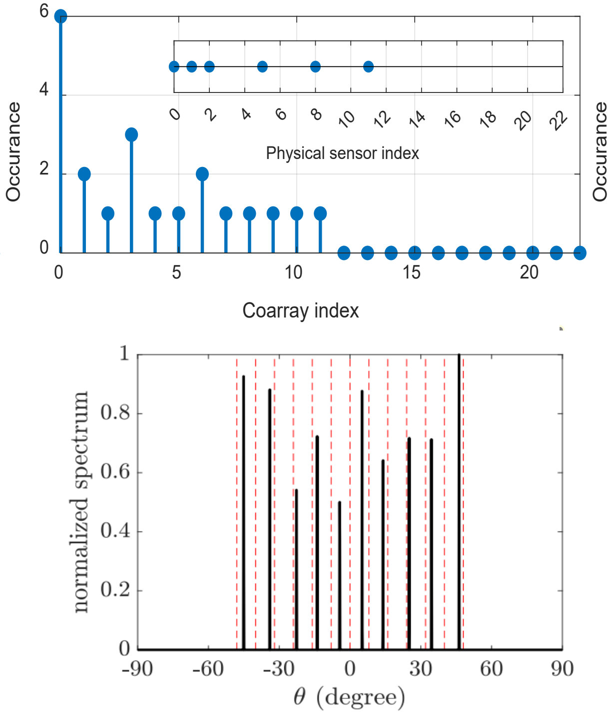


RMSE for two closely spaced source case

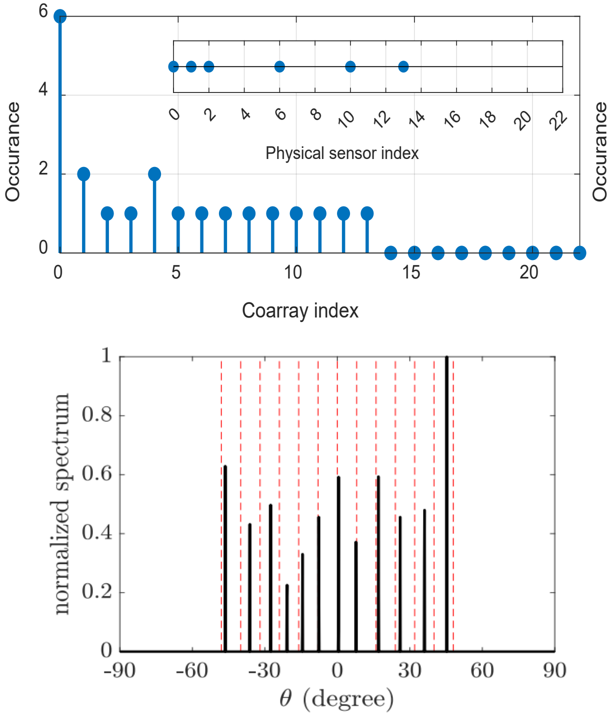
Coprime array (9 lags; max lag 9)



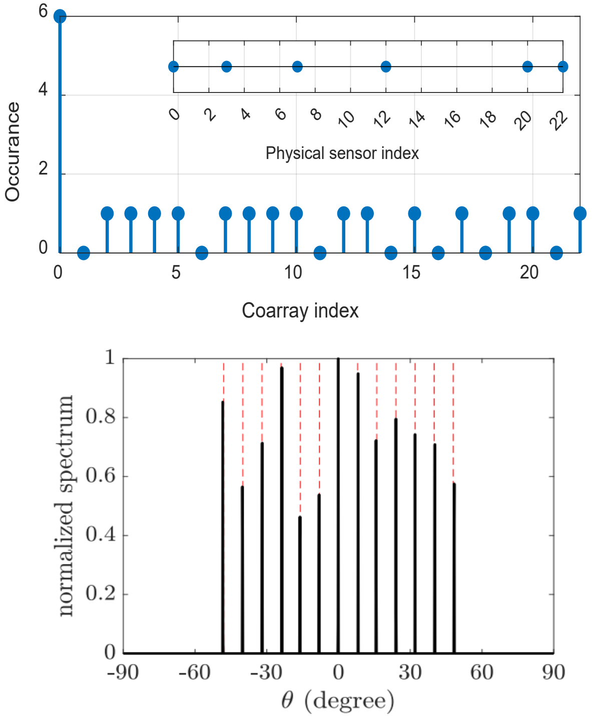
Nested array (12 lags; max lag 11)



MISC array (14 lags; max lag 13)



ONRA (16 lags; max lag 22)



A. Ahmed and Y. D. Zhang, "Generalized non-redundant sparse array designs," *IEEE Trans. Signal Processing*, 2021.

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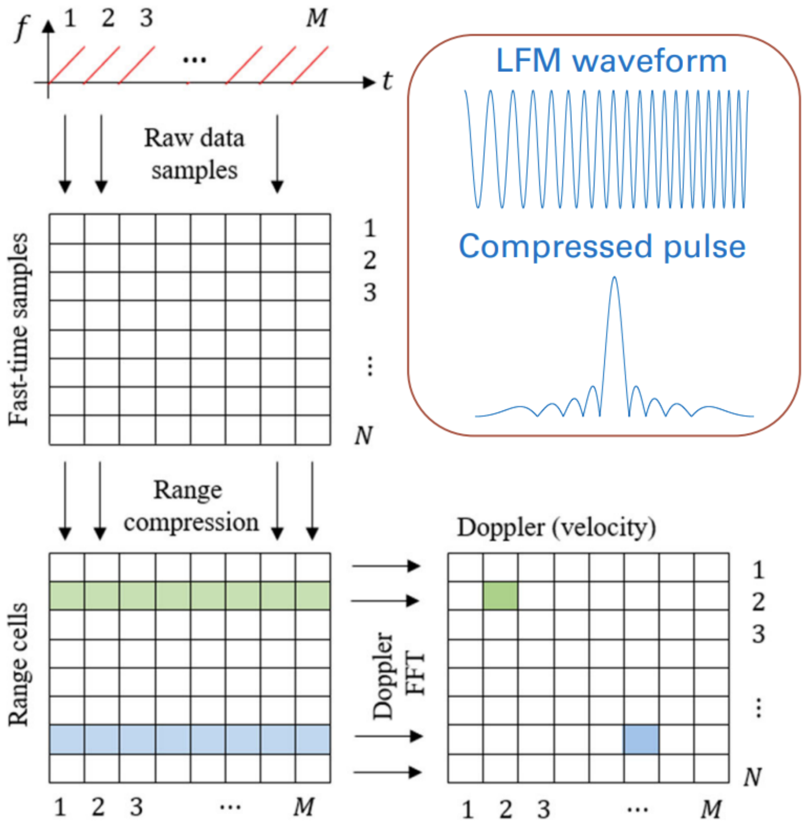
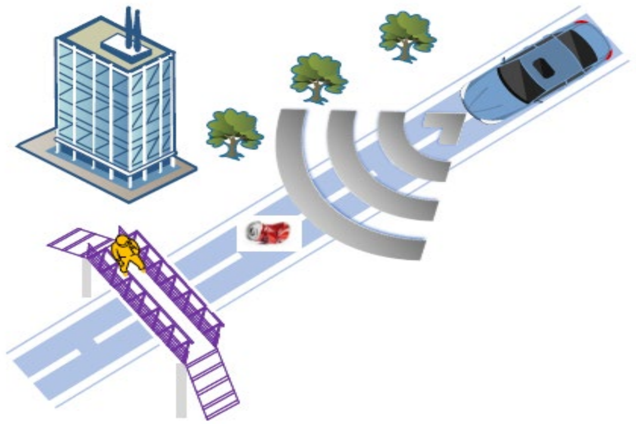
Automotive radar application

- Radar has emerged as one of the key technologies in autonomous driving systems.
  - Low-cost implementation
  - Resilient sensing in all weather/lighting conditions
- Automotive radar first performs range Doppler mapping, and the result data may only provide few (even one) data samples.
- A large aperture in both azimuth and elevation is important to identify objects and enable drive-over and drive-under.
- To achieve a $\Delta\theta = 1^{\text{o}}$ resolution, a 2D array with an aperture of $D = 1.4/(\pi \sin(\Delta\theta/2)) \approx 51$ wavelengths is needed in each dimension.
- Very few antennas can be used to keep a low cost.



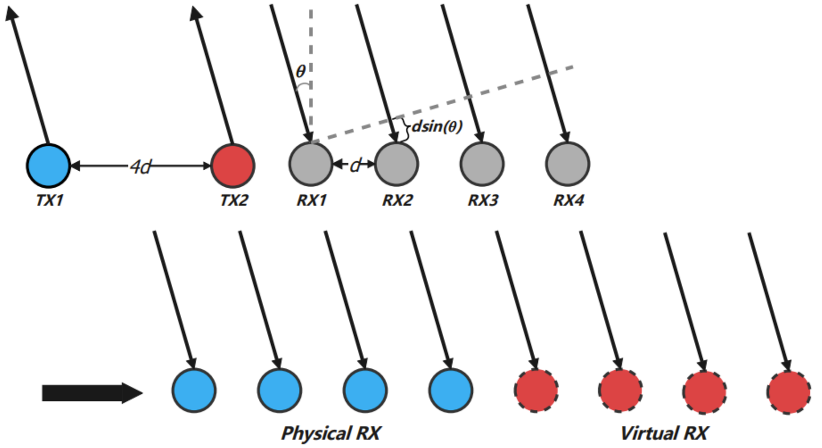


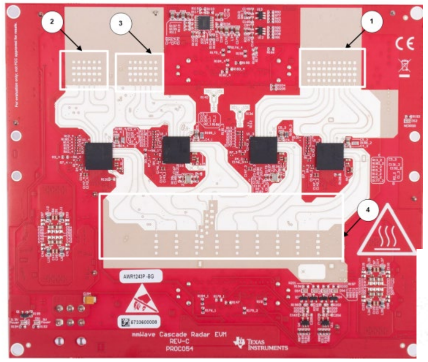
S. Sun and Y. D. Zhang, “4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach,” *IEEE Journal of Selected Topics in Signal Processing*, 2021.

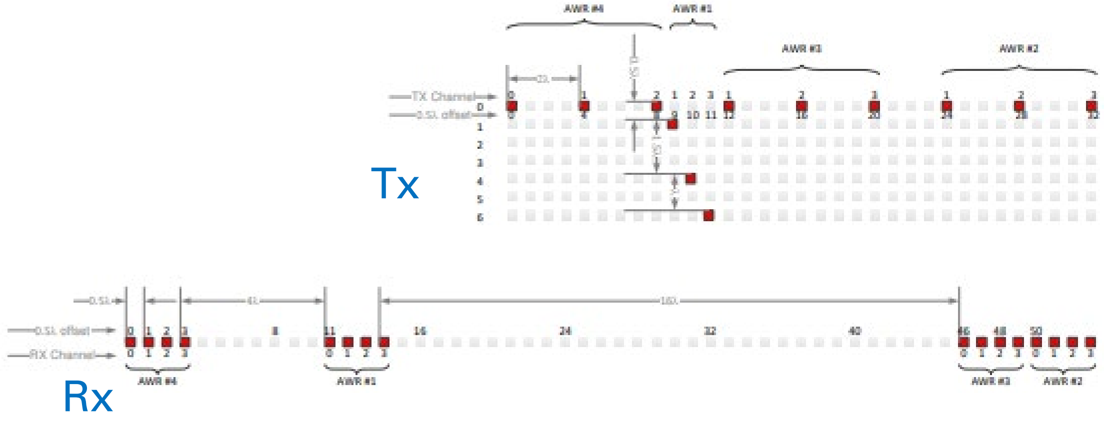
S. Sun and Y. D. Zhang, "Redefining radar perception for autonomous driving: The role of sparse array and waveform design in 4D automotive radar," IEEE Signal Processing Society Webinar Series, Sept. 2023.

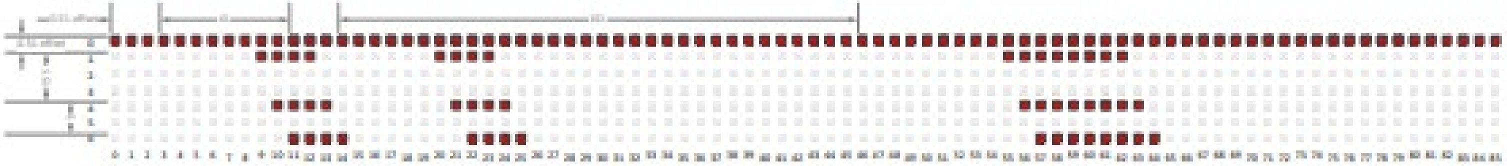
MIMO radar and sum coarray

- Sparse 2-D MIMO radar is commonly used to achieve sum coarray.
- The sum coarray in a MIMO radar is synthesized as $S = \{x + y \,|x \in \mathbb{S}_T, y \in \mathbb{S}_R\}$, where $\mathbb{S}_T$ and $\mathbb{S}_R$ are Tx and Rx antenna positions.
- Texas Instruments (TI) AWRx Cascaded Radar RF Evaluation Module (MMWCAS-RF-EVM) use 12 Tx 16 Rx configuration to provide a large horizontal sum coarray and small vertical aperture.









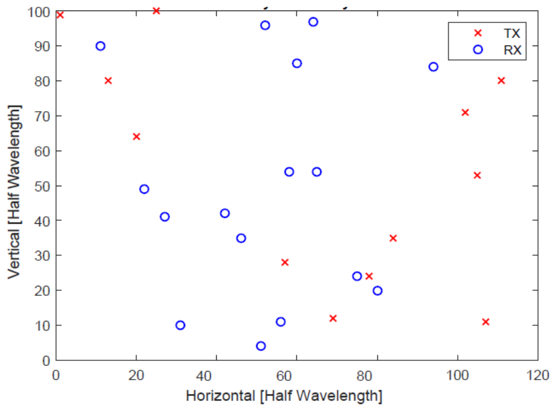
Sum coarray

S. Sun and Y. D. Zhang, "4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach," *IEEE Journal of Selected Topics in Signal Processing*, 2021.

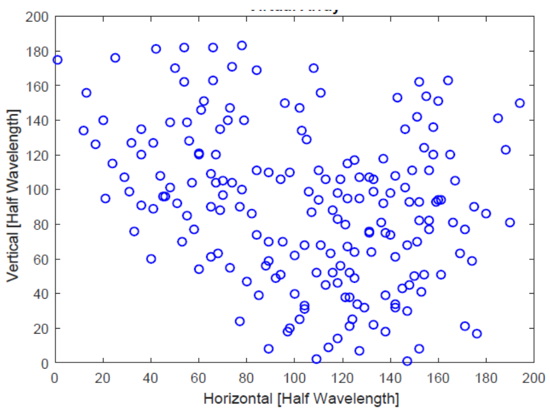
Automotive radar application

Simulation example:

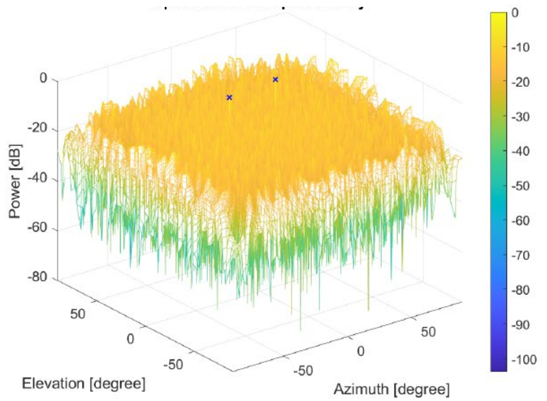
- Consider randomly placing 12 Tx and 16 Rx antennas in about 100 half-wavelength 2D range with 196 virtual antennas: Very sparse antenna placement
- Direct 2-D imaging renders high sidelobes due to missing elements in the sum coarray
- Data completion is important to recover a uniform rectangular array (URA) and reduce the sidelobes
- How to perform data completion when there is a single snapshot?

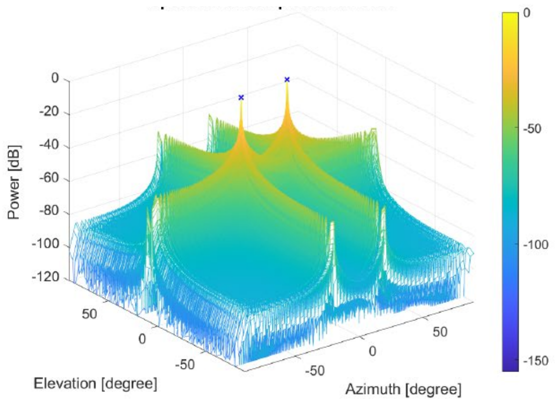


Physical antennas



virtual sensors of sum coarray





Azimuth & elevation imaging without completion and with completion

S. Sun and Y. D. Zhang, “4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach,” *IEEE Journal of Selected Topics in Signal Processing*, 2021.

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Automotive radar application

Hankel matrix construction for M-element ULA:

- Assume noiseless array response $\boldsymbol{y} = [y_1, y_2, \dots, y_M]^T$, we construct a Hankel matrix as

$$\mathcal{H}(\boldsymbol{y}) = \begin{bmatrix} y_1 & y_2 & \cdots & y_L \\ y_2 & y_3 & \cdots & y_{L+1} \\ y_3 & y_4 & \cdots & y_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{M_1} & y_{M_1+1} & \cdots & y_M \end{bmatrix}$$

  where L is the pencil parameter and $M_1 = M - L + 1$

- When K ($K < M_1$ and $K < L$) sources imping to the array, the Hankel matrix $\mathcal{H}(\boldsymbol{y})$ has a Vandermonde decomposition structure $\mathcal{H}(\boldsymbol{y}) = \boldsymbol{A}\,\boldsymbol{\Sigma}_s\boldsymbol{B}^T$ with rank K, where

$$\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \cdots, \boldsymbol{a}(\theta_K)] \text{ with } \boldsymbol{a}(\theta_k) = \left[1, e^{\frac{j2\pi d \sin(\theta_k)}{\lambda}}, \cdots, e^{\frac{j2\pi (M_1-1) d \sin(\theta_k)}{\lambda}}\right]^T$$

$$\boldsymbol{B} = [\boldsymbol{b}(\theta_1), \cdots, \boldsymbol{b}(\theta_K)] \text{ with } \boldsymbol{b}(\theta_k) = \left[1, e^{\frac{j2\pi d \sin(\theta_k)}{\lambda}}, \cdots, e^{\frac{j2\pi (L-1) d \sin(\theta_k)}{\lambda}}\right]^T$$

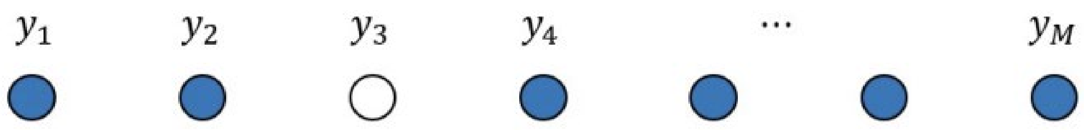
$$\boldsymbol{\Sigma}_s = \mathrm{diag}([\beta_1, \cdots, \beta_K])$$

S. Sun and Y. D. Zhang, “4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach,” *IEEE Journal of Selected Topics in Signal Processing*, 2021.

S. Sun and Y. D. Zhang, "Redefining radar perception for autonomous driving: The role of sparse array and waveform design in 4D automotive radar," IEEE Signal Processing Society Webinar Series, Sept. 2023.

Automotive radar application

Hankel matrix completion for sparse linear array:



- Missing sum coarray elements render a Hankel matrix with missing elements:

$$\mathcal{H}(\boldsymbol{y}) = \begin{bmatrix} y_1 & y_2 & \cdots & y_L \\ y_2 & y_3 & \cdots & y_{L+1} \\ y_3 & y_4 & \cdots & y_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{M_1} & y_{M_1+1} & \cdots & y_M \end{bmatrix}$$

- The forward-only Hankel matrix completion problem is to find a Hankel matrix $\mathcal{H}(\boldsymbol{y})$ that has a minimum rank and its distance to the original data matrix at the observed positions meets the required error bound δ:

$$\min_{\boldsymbol{x}} \ \operatorname{rank}\big(\mathcal{H}(\boldsymbol{x})\big)$$
$$\text{s.t.} \ \|\mathcal{H}(\boldsymbol{x}) \circ \boldsymbol{M} - \mathcal{H}(\boldsymbol{y})\|_F \leq \delta$$

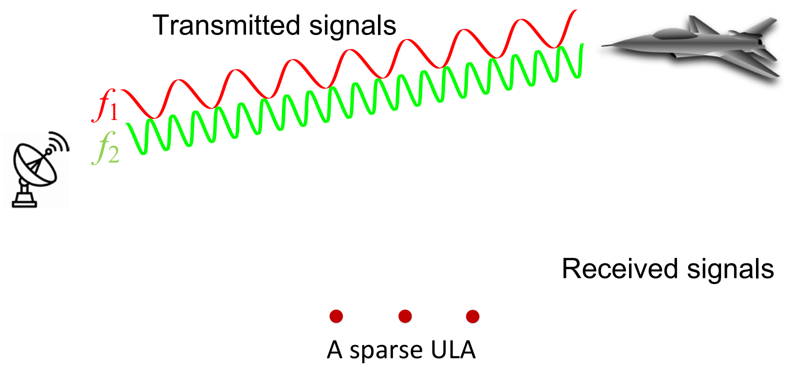
where $\boldsymbol{M}$ is a mark matrix.

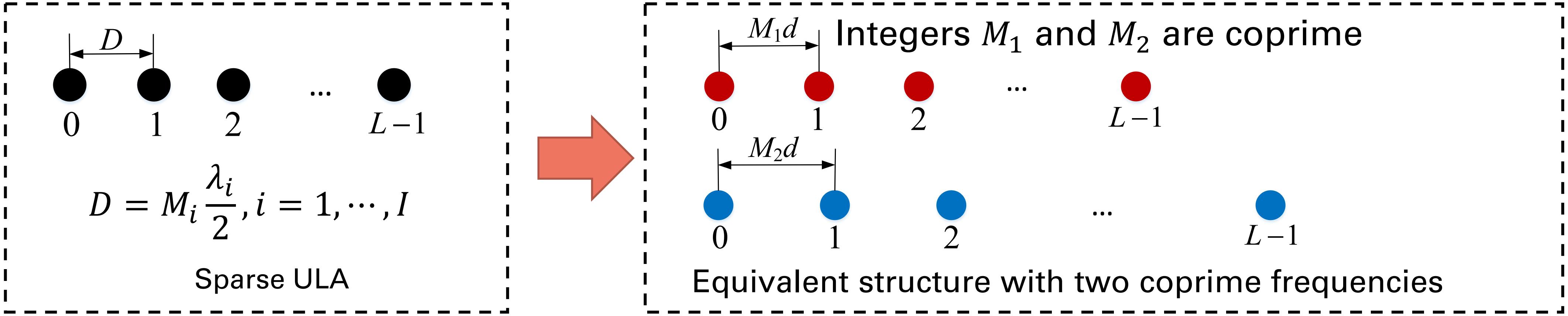
S. Sun and Y. D. Zhang, “4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach,” *IEEE Journal of Selected Topics in Signal Processing*, 2021.

S. Sun and Y. D. Zhang, "Redefining radar perception for autonomous driving: The role of sparse array and waveform design in 4D automotive radar," IEEE Signal Processing Society Webinar Series, Sept. 2023.

Multi-frequency sparse array

- Difference coarray is obtained from array data covariance matrix, which requires **time-domain snapshots**.
- Can we further utilize resources in other domains, such as **frequency**?
- Multi-frequency/frequency-switching sparse arrays exploit two or more frequencies to obtain virtual arrays: **Trade frequency resource to spatial resources.**

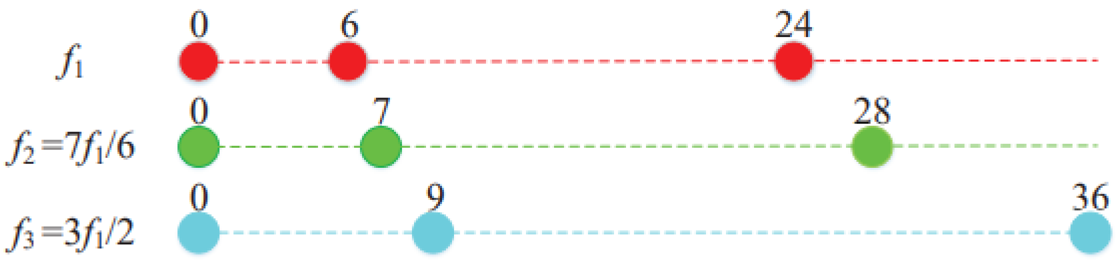




S. Qin, Y. D. Zhang, M. G. Amin, and B. Himed, "DOA estimation exploiting a uniform linear array with multiple co-prime frequencies," *Signal Processing*, 2017.

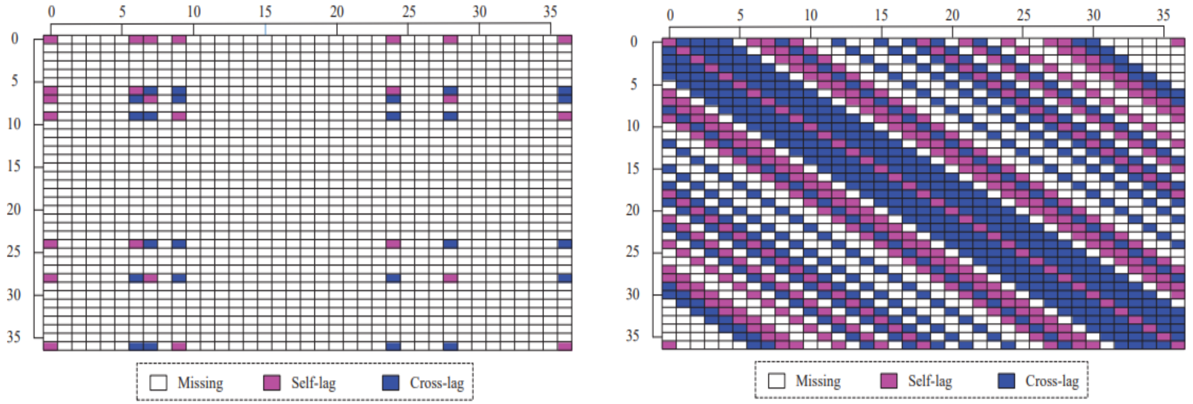
Multi-frequency sparse array

Maximize the number of virtual sensors: choosing the array configuration and frequencies such that the virtual sensors corresponding to different frequencies do not overlap.



Example: 3 antennas, 3 frequencies

- 7 virtual sensors at $\{0,6,7,9,24,28,36\}\bar{d}$, where $\bar{d}$ denotes half-wavelength without referring to a specific frequency
- 10 non-negative self-lags:
  12 non-negative cross-lags:
- All lags appear only once (redundancy-free)



More detailed on multi-frequency sparse array to be discussed in
Keynote: Harnessing Frequency Diversity for Enhanced Direction-of-Arrival Estimation
Third Workshop on Signal Processing for Autonomous Systems
held in conjunction with EUSIPCO 2025, on September 12

S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "Enhanced DOA estimation exploiting multi-frequency sparse array," *IEEE Trans. Signal Processing*, 2021.

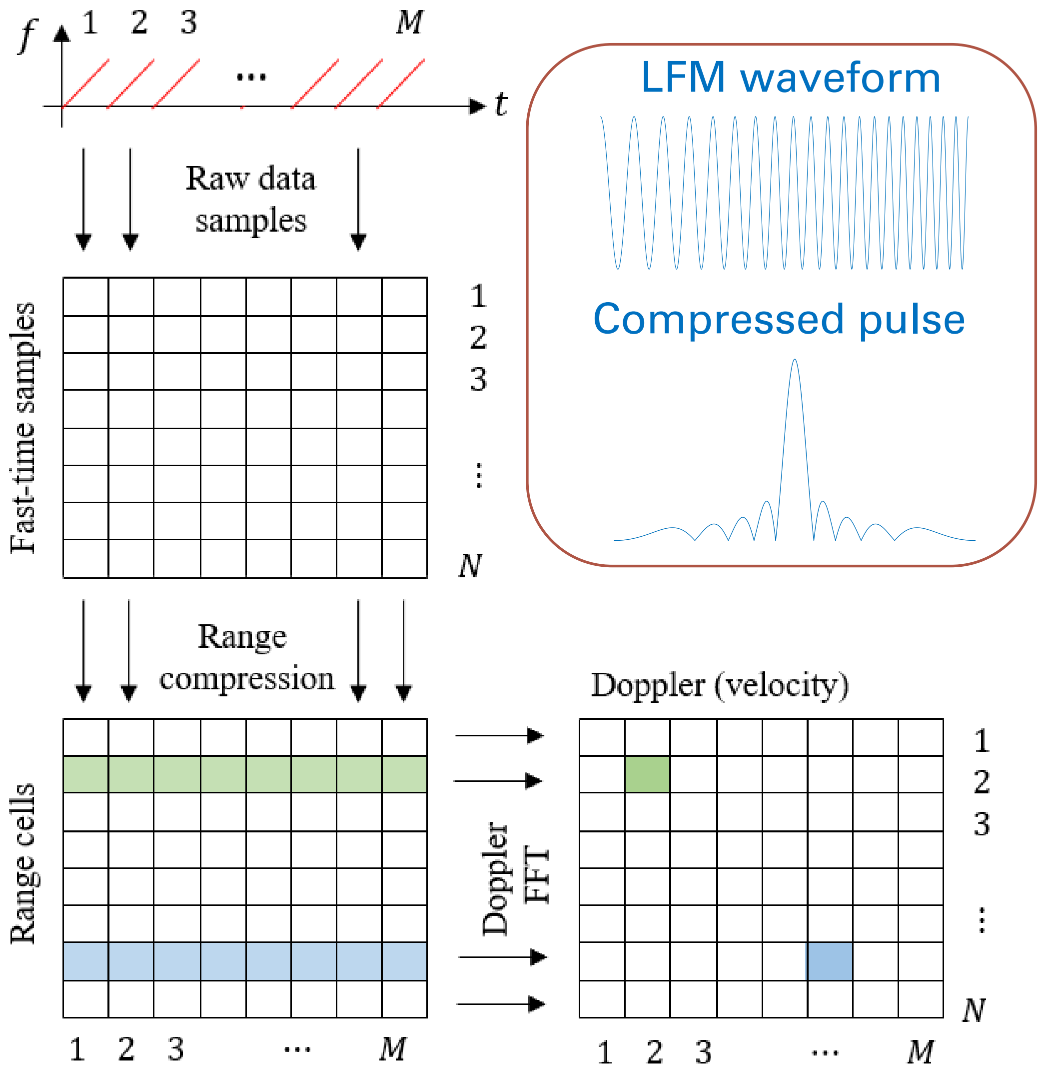
Part III: Sparse Waveform Design and Processing

- Sparse waveforms in slow-time and step frequencies
- Shared waveform design via integer partitioning

Radar operation

Typical radar operations:

- Periodically transmit same probing waveforms
- Matched filtering of received signal waveforms provides range information
- Fourier transform of slow-time data yields target Doppler (velocity) information



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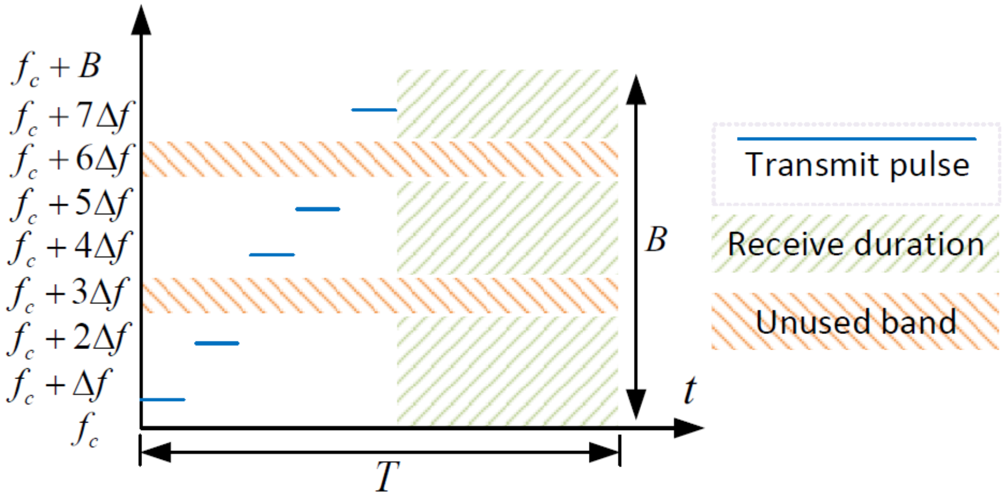
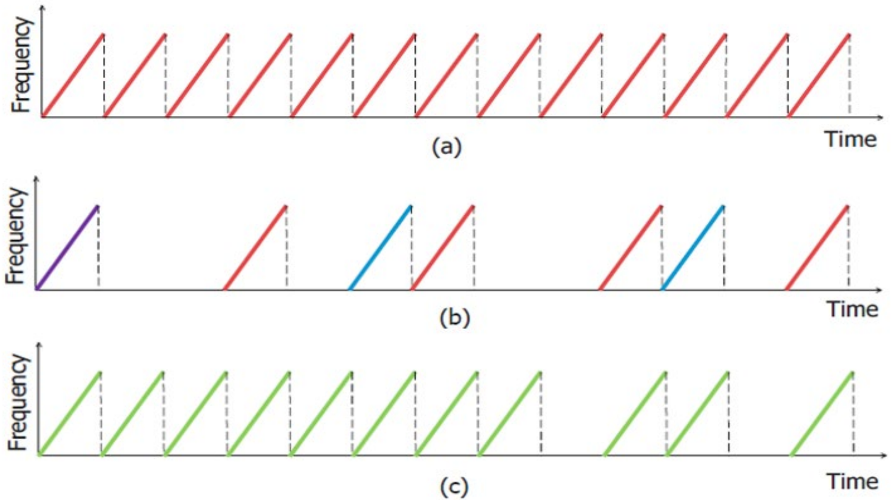
Sparse time-domain sampling for spectrum estimation

- Spectrum usage becomes increasingly congested.
- Traditional radar waveforms like frequency-modulated continuous-wave (FMCW) signals offer high-resolution range and Doppler estimation but consume substantial time-frequency resources.
- Sparse waveform design strategically minimizes time-frequency occupancy while preserving high-resolution sensing.
- We explore sparse step-frequency and slow-time pulse designs as well as sparsity-based range-Doppler processing.





S. Sun and Y. D. Zhang, "4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach," IEEE Journal of Selected Topics in Signal Processing, 2021.

L. Xu, S. Sun, K. V. Mishra, and Y. D. Zhang, "Automotive FMCW radar with difference co-chirps," IEEE Trans. Aerospace and Electronic Systems, 2023.

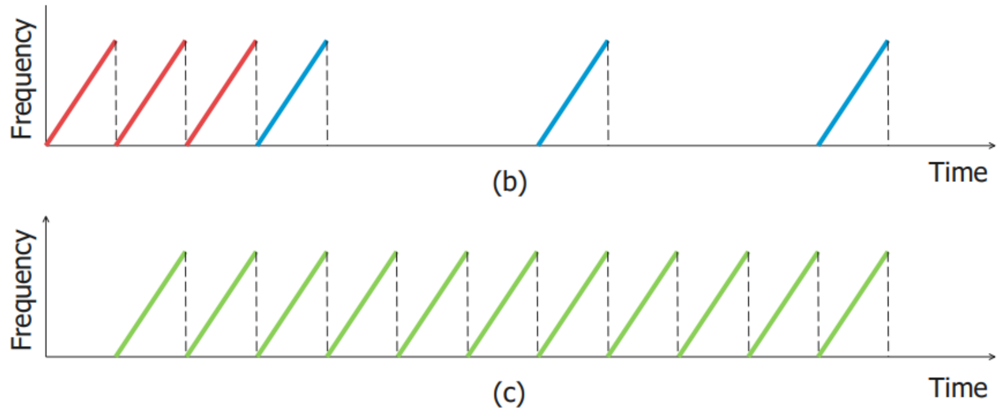
A S P Lab

Difference co-chirp-based FMCW radar

- Consider a uniformly spaced chirp set $\mathbb{S} = \{m_1, m_2, \dots, m_M\}$, with m_i describing the position of the i-th chirp.
- Difference co-chirp set

$$\mathbb{S}_{\text{diff}} = \{m_i - m_j\}, \forall i, j \in \mathbb{S}$$

- For FMCW radar with nested chirps, : We now examine the FMCW radar that schedules its slow-time emission following the nested chirp relationship.
- Similar defined for co-prime chirps.
- Offerings:
    - High Doppler resolution with few chirps
    - Significantly reduced interference to victim radars
    - Support multiple radars simultaneous transmission with low interference



Nested chirps and lags

Difference co-chirp-based FMCW radar

- Assuming K signals in the scene, the noise-free received signal $x(m,t)$ corresponding to the m-th pulse is expressed a

$$x(m,t) = \mathrm{rect}\left(\frac{t - mT_p}{T}\right) e^{j2\pi\left[f_c + \frac{B}{T}(t - mT_p)\right](t - mT_p)]}$$

- Directly obtain Doppler from sparse pulses would result in high sidelobes.
- To perform co-chirp processing, we need to obtain a high number of “snapshots” to construct the second-order covariance matrix.
- In many applications such as automotive radar, the Doppler shift during fast-time sampling of a single chirp can be considered constant.
- We treat the Q fast-time samples as “snapshots” for sample Doppler covariance matrix construction:

$$\widehat{\boldsymbol{R}}_{\text{nested}} = \frac{1}{Q}\sum_{q=1}^{Q} \boldsymbol{x}_{\text{nested}}^{q}\left(\boldsymbol{x}_{\text{nested}}^{q}\right)^{H}$$

- By vectorizing $\widehat{\boldsymbol{R}}_{\text{nested}}$, the co-chirp signal model is $\hat{\boldsymbol{r}}_{\text{nested}} = \mathrm{vec}(\widehat{\boldsymbol{R}}_{\text{nested}})$.

Sparse time-domain sampling for spectrum estimation

How to support multiple sensing platforms in a fair way?

We develop a novel strategy to partition a one- or multi-dimensional consecutive integer number set into multiple identical, possibly rotated, subsets.

The obtained subsets are required to have consecutive difference lags which are desired to be as long as possible.

The proposed technique first exploits one-dimensional nested subsets, and the results are extended to achieve two- and multi-subset partitioning as well as in two- and multi-dimensional spaces.

The results are useful to various sensing and communication applications. Sparse step-frequency waveform design for range estimation in automotive radar is demonstrated as an example.

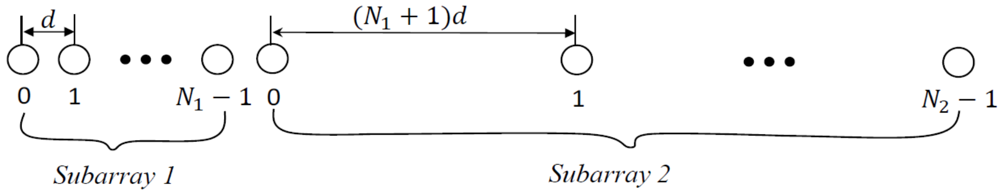
Sparse design: From a single platform to multiple platforms

Many sparse array configurations have been developed.

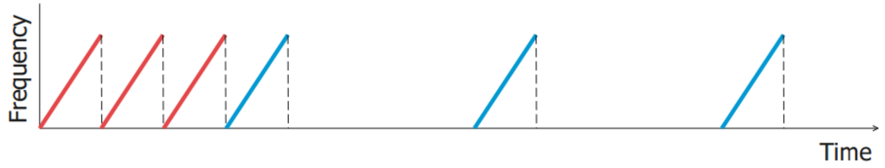
N antennas can be used to achieve $O(N^2)$ lags.

Nested array family is a poplar choice to achieve consecutive lags [1].

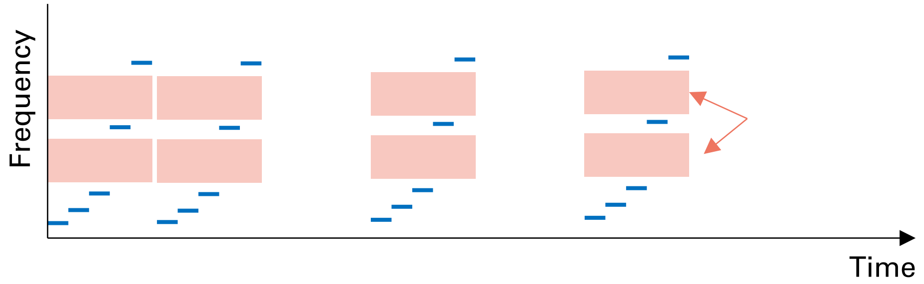
This concept has also been adopted in radar waveform design, such as using nested slow-time pulses and fast-time step frequencies [2, 3].



Nested sparse arrays



Nested slow-time pulses



Sparse step frequencies and slow-time pulses

From single-radar design to multi-radar design

- Existing array processing and waveform design schemes only consider a single radar platform.
- Giving a uniform 1-D or 2-D array, uniform slow-time pulses, and/or uniform step frequencies, can we partition them into multiple ***identical subsets***, each achieving ***consecutive lags***, so that they ***support multiple radars*** without mutual interference?

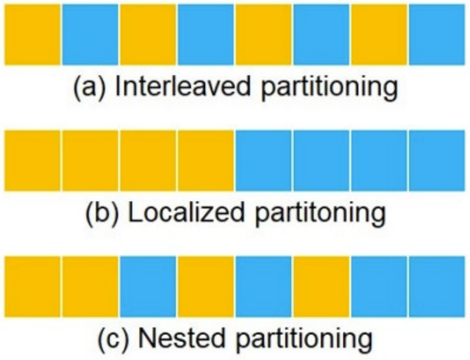
[1] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Processing*, 2010.
[2] S. Sun and Y. D. Zhang, "4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach," IEEE J. Selected Topics in Signal Processing, 2021.
[3] L. Xu, S. Sun, K. V. Mishra, and Y. D. Zhang, "Automotive FMCW radar with difference co-chirps," IEEE Trans. Aerospace and Electronic Systems, 2023.

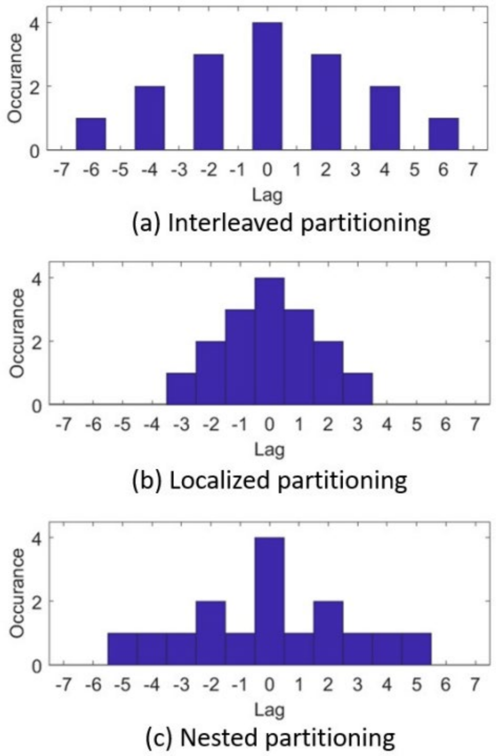
Identical partitioning of integer set

- We address this problem by first considering the identical partitioning of an integer set.
- Let $\mathbb{Q} = \{1, \cdots, Q\}$ be a 1-D set of Q continuous integers, where Q is an even integer.
- The objective is to partition $\mathbb{Q}$ into $G \geq 2$ subsets, i.e., $\mathbb{Q} = \mathbb{Q}_1 \cup \cdots \cup \mathbb{Q}_G$ such that each subset $\mathbb{Q}_1, \mathbb{Q}_2, \cdots, \mathbb{Q}_G$
  - has an identical, possibly rotated (flipped), pattern
  - has consecutive difference lags, as long as possible

Example

- Partition consecutive set $\mathbb{Q} = \{1, \cdots, 8\}$ into $G = 2$ subsets.
  - **Interleaved partitioning:** uniform undersampling pattern leads to equally spaced gaps in the difference lags and causing alias
  - **Localized partitioning:** provides consecutive lags, but the lags only extend between −3 and 3
  - **Nested partitioning:** a preferred choice as it provides consecutive lags between −5 and 5





Y. D. Zhang and S. Sun, "Identical partitioning of consecutive integer set," IEEE SAM Workshop, 2024.

Identical partitioning of integer set

- For two-subset partitioning, the first nested subset consists of 2 consecutive elements as inner group, followed by an arbitrary number (N) of outer group elements separated by 2.
- For each subset with $N+2$ elements, their locations are given as

$$\{1,2,4,6,\cdots,2N,2N+2\} \text{ and } \{3,5,\cdots,2N-1,2N+1,2N+3,2N+4\}$$

- Each subarray offers consecutive lags between $-2N-1$ and $2N+1$.

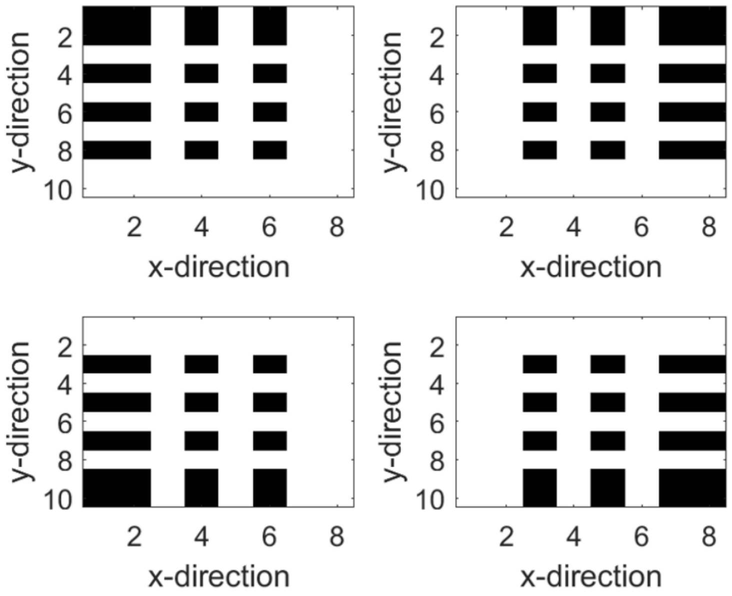
$N=3$

Multi-dimensional partitioning

- We can extend two-subset 1-D partitioning results into a multi-dimensional case, illustrated using a 2-D example.
- Outer product of $\boldsymbol{m}_{g_1}^{[1]}$ and $\boldsymbol{m}_{g_2}^{[2]}$ with $g_1, g_2 = 1,2$:

$$\boldsymbol{M}_{g_1,g_2} = \boldsymbol{m}_{g_1}^{[1]}\left(\boldsymbol{m}_{g_2}^{[2]}\right)^{\mathrm{T}}$$

- Partitioned subsets can be used for 2-D array or array/waveform combinations.



Example:

- For N_1 = 2 and N_2 =3, each subset forms the same continuous difference lag set in [−5 : 5, −7 : 7].

Identical partitioning of integer set

Multi-layer 1-D partitioning

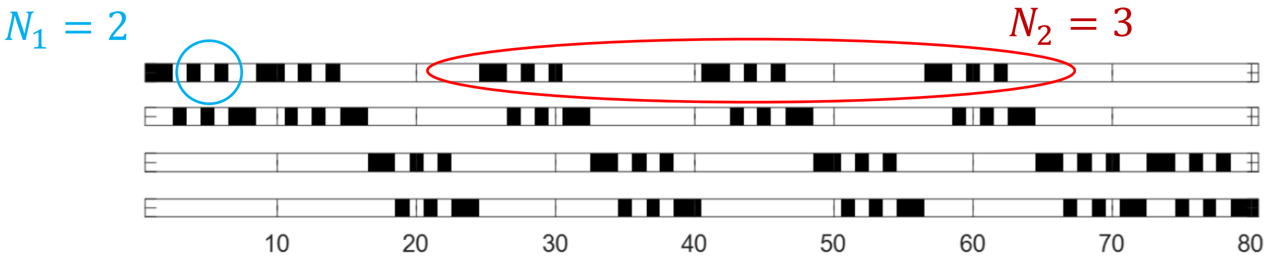
- Vectorizing the masking matrix or tensor into a vector, we obtain a multi-layer partitioning scheme where the partitioning pattern is the Kronecker product of D subsets ($D \geq 2$):

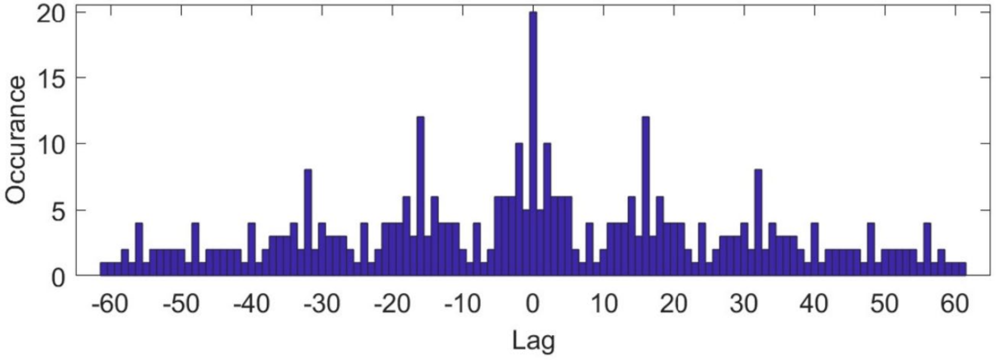
$$\widetilde{\boldsymbol{m}}_{g_1,\ldots,g_D} = \boldsymbol{m}_{g_D} \otimes \cdots \otimes \boldsymbol{m}_{g_2} \otimes \boldsymbol{m}_{g_1}$$

with $g_1, g_2, \cdots, g_D = 1, 2$, yielding $G = 2^D$ identical subsets.

Example:

- $D = 2$ lays with $N_1 = 2$ and $N_2 = 3$
- 4 identical subsets with same 61 consecutive lags in each subset
- More subsets are obtained with less lag redundancy

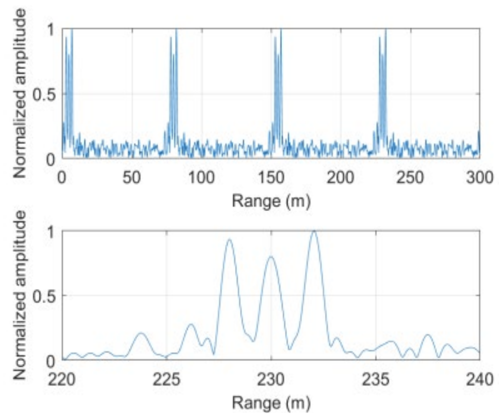




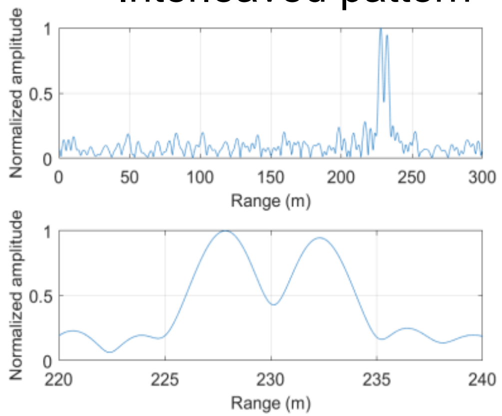
Such results can be extended to multi-dimensional case.

Application example: Range estimation

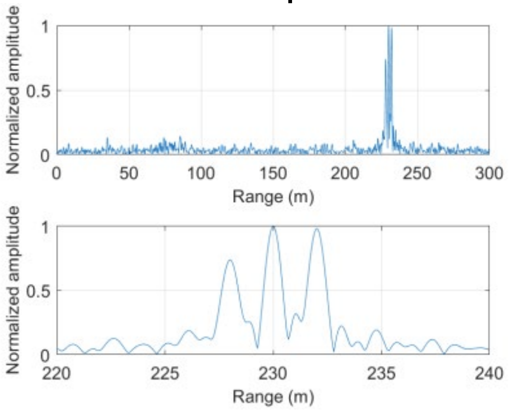
- Consider an example that performs range estimation exploiting sparse frequencies.
- 3 targets with ranges of 227 m, 230 m, and 232 m.
- Step-frequency continuous-wave (SFCW) radar where $B = 200$ MHz bandwidth is divided into 400 frequency bins ($f_\Delta = 0.5$ MHz).
- For a single radar unit, the range resolution is ΔR = 0.75 m, and the maximum unambiguous range is $R_{\max}$ = 300 m.
- When the 400 step frequencies are partitioned into 4 orthogonal subsets to allow 4 radars to operation without interference, each subset utilizes 100 step frequencies:
  - **Interleaved partitioning:** The step frequency is increased to $4f_\Delta = 2$ MHz, reducing unambiguous range to 75 m.
  - **Localized partitioning:** The bandwidth is reduced to $B/4$, thus compromising the range resolution to 3 m.
  - **Proposed super-nested portioning:** The unambiguous range remain the same, and the maximum frequency span of each waveform is 357 step frequencies, rendering range resolution to be 0.84 m (a 12% degradation).



Interleaved pattern



Localized pattern



Partitioning pattern

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Identical division of integer set

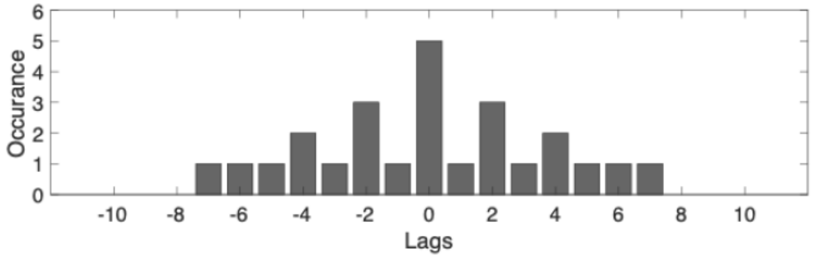
- To provide higher design flexibility with improved efficiency, we consider identical division.
- The difference between division and partitioning is that not all elements are used in the division scheme.
- For identical division of subsets with core elements separated by 3, the positions are given as

  $\{1, 3, 6, 9, \cdots, 3N - 3, 3N, 3N + 1\}$

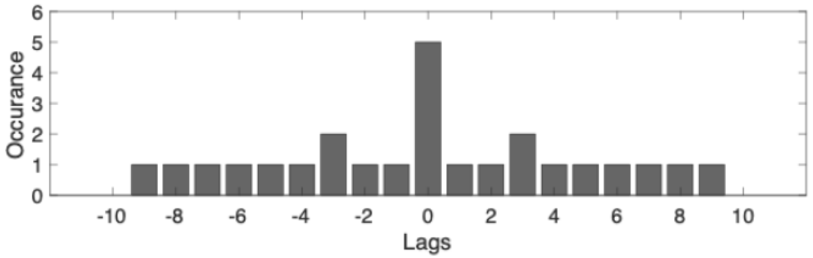
  $\{4, 5, 8, 11, \cdots, 3N - 1, 3N + 2, 3N + 4\}$
- Both subarrays have consecutive difference lags between $-3N$ and $3N$.
- For the same number of subset elements, the identical division scheme provides $N - 1$ additional lags in each side compared to identical partitioning.

(a) Nested partition (showing for $N = 3$)

(b) Identical division (showing for $K = 1$ and $N = 3$)



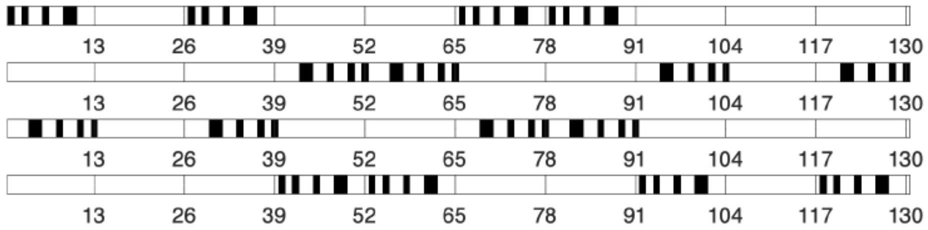
(a) Weight function of difference lags from nested partition



A. Namboothiri, M. W. Chowdhury, and Y. D. Zhang, "Non-overlapping identical division of integer set," Asilomar Conference on Signals, Systems, and Computers, 2024.

Identical division of integer set

- We can similarly form multilayer division schemes.
- In a 2-layer ($D = 2$) case, 4 subsets are obtained, and the number of consecutive lags of each subset is increased to $(3N_1)(3N_2 + 4) + (3N_2)$ from $(2N_1 + 4)(2N_2 + 1) + (2N_1 + 1)$.

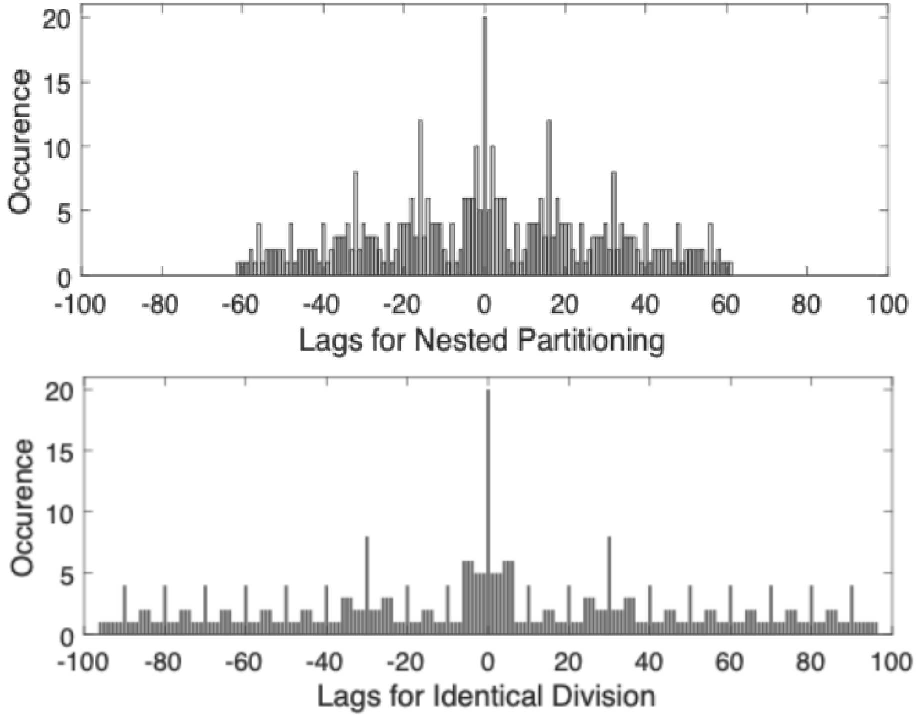


(a) Four subset identical division of a consecutive integer set.

(b) Union set of the four identically divided subsets

Example

- When the same number of 20 subset elements are used as the previous example, where $N_1 = 2$ and $N_2 = 3$, a subset in the identical division scheme achieves 96 consecutive lags, compared to 61 in the identical partition case.
- The occurrence graph (histogram of lags) clearly shows reduced redundancy in the identical division.



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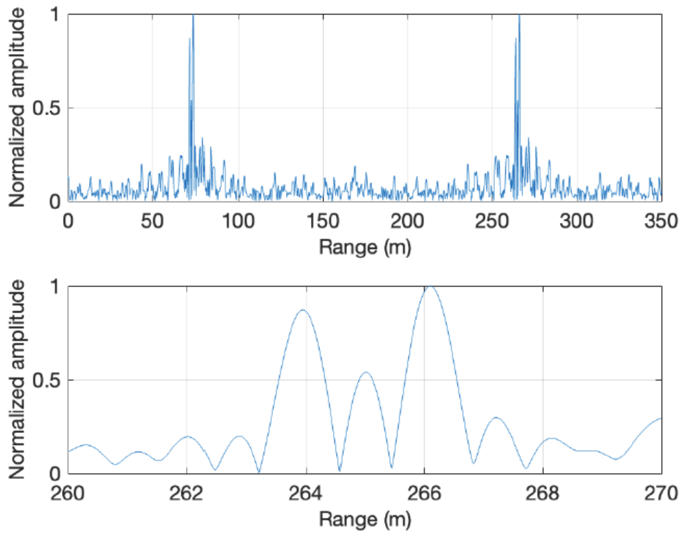
Application example: Range estimation

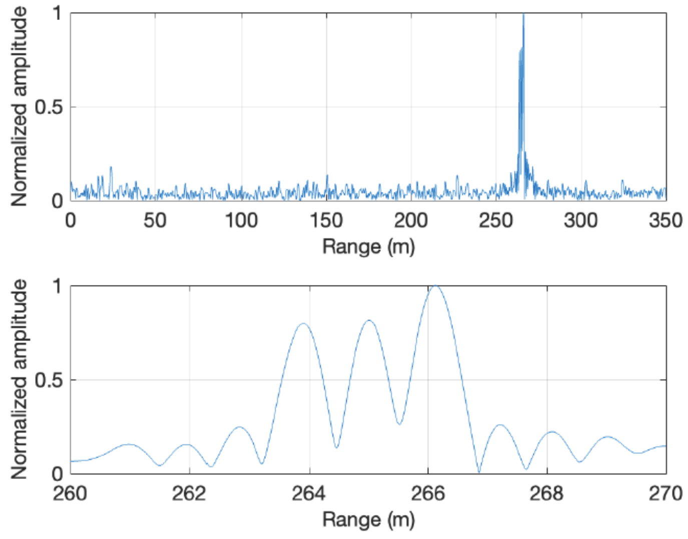
- Consider a step-frequency radar with available bandwidth $B = 200$ MHz.
- Three targets are located at ranges of 264 m, 265 m, and 266 m.
- $D = 2$ layers, providing 4 radars with interference-free subsets of step frequency bins.



Identical partition

Identical partition

- $N_1 = N_2 = 6$
- $N_f = 256$ frequency bins with $\Delta f = 781.25$ kHz.
- $R_{\max} = 192$ m: Not enough to provide unambiguous range estimation



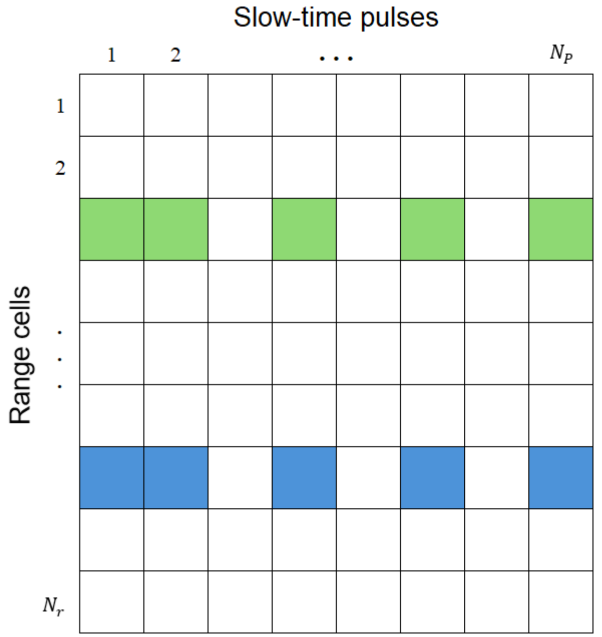
Identical division

Identical division

- $N_1 = N_2 = 6$
- $N_f = 484$ frequency bins with $\Delta f = 413.22$ kHz.
- $R_{\max} = 363$ m: enable unambiguous range estimation

Application example: Range-Doppler estimation

- By exploiting fast-time data samples to compute second-order covariances, consecutive slow-time lags are obtained from the sparsely partitioned slow-time pulses.
- For FMCW radar used for short-range target detection, all fast-time data may be used to compute the autocorrelation function for slow-time data.
- For pulse radar waveforms with a low duty cycle, the autocorrelation function can be computed based only on the fast-time data associated with the range cells where targets are detected in target range estimation.
  - Reduced computational complexity as fewer range cell data are used
  - Reduced noise effect as target-free fast-time samples are excluded
  - Does not require range-Doppler association because Doppler estimation is separately performed for each range cell



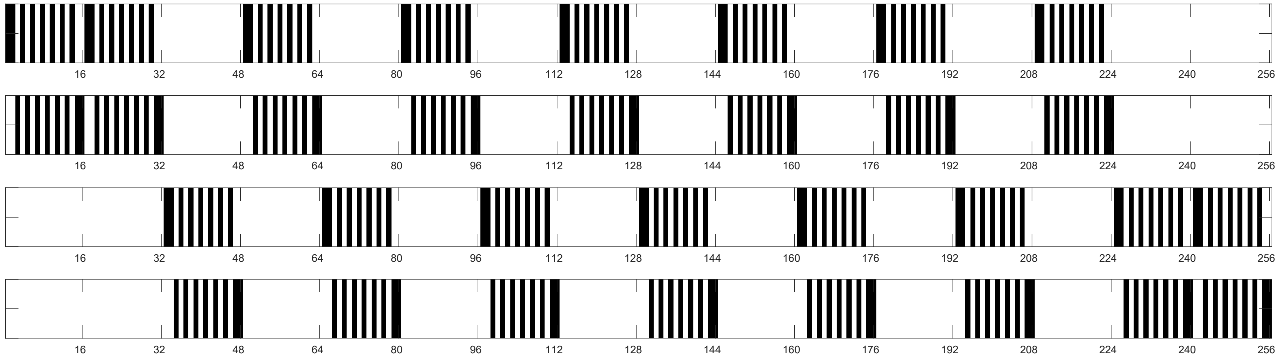
L. Xu, S. Sun, K. V. Mishra, and Y. D. Zhang, "Automotive FMCW radar with difference co-chirps," IEEE Trans. Aerospace and Electronic Systems, 2023.

M. W. T. S. Chowdhury, Y. D. Zhang, and B. Himed, "Sparse radar waveform design exploiting consecutive integer set partitioning," IEEE Int. Radar Conf., 2025.

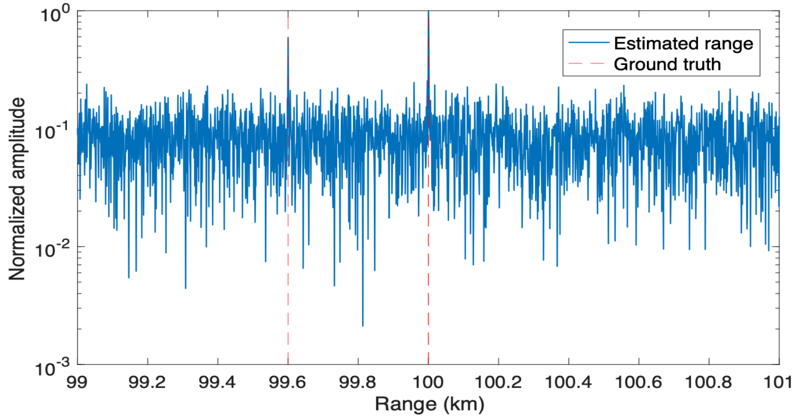
Application example: Range-Doppler estimation

- 256 slow-time slots in a CPI, which are divided into 4 super-nested slow-time pulse subsets.

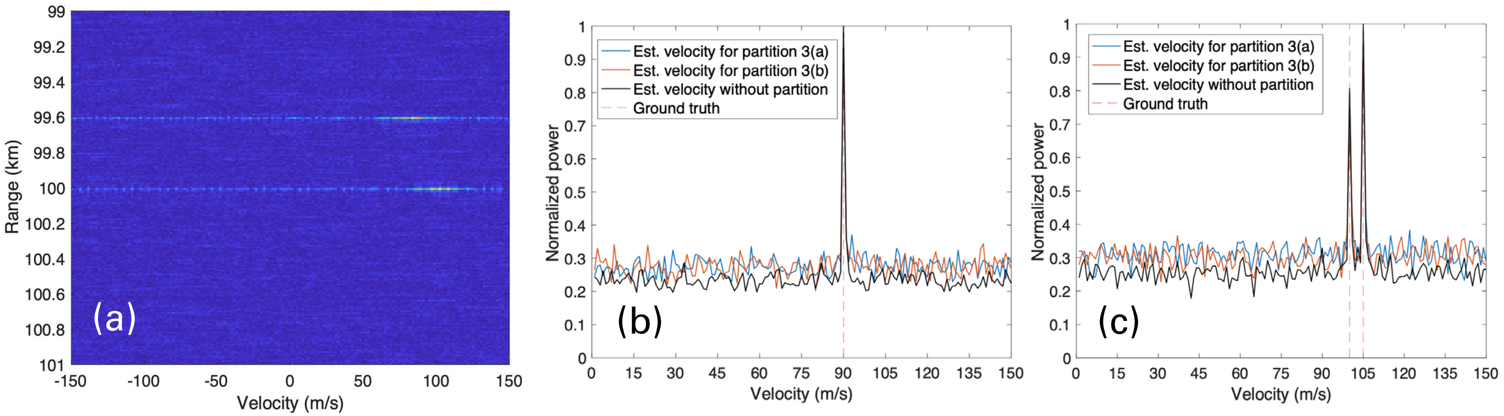
- Three targets are present:
  - Two targets at a range of $100\ \mathrm{km}$ with radial velocities of $100\ \mathrm{m/s}$ and $105\ \mathrm{m/s}$
  - Another target at $99.6\ \mathrm{km}$ with radial velocity of $90\ \mathrm{m/s}$
  - The input SNR of the raw data is $-30\ \mathrm{dB}$ for all targets.



Division of slow-time samples



Range spectrum of the three targets



Doppler estimation using 2D FFT and partitioning

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Sparse array design and processing: What is next?

Sparse Array Design and Processing

A: More consecutive lags and lower coupling

B. Sparsity-based DOA estimation

C. Structured matrix completion for DOA estimation

Sparse Waveform Design and Processing

A: Sparse waveforms in slow-time and step frequencies

B. Shared waveform design via integer partitioning

General issues

- Low-complexity implementations
- Performance and bound analysis
- Robustness issues

Two/multi-dimensional arrays

- Array design
- Extreme sparse arrays
- Tensor-based processing

Bandwidth exploitation

- DOF analysis for wideband signals
- Low-complexity solutions
- Fractional sparse arrays

Signal coherency

- Coherent/correlated signals
- Mixed uncorrelated/coherent signals

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Sparse waveform design and processing: What is next?

Sparse Array Design and Processing

A: More consecutive lags and lower coupling

B. Sparsity-based DOA estimation

C. Structured matrix completion for DOA estimation

Sparse Waveform Design and Processing

A: Sparse waveforms in slow-time and step frequencies

B. Shared waveform design via integer partitioning

General issues

- More generalized waveform solutions
- Real-world requirements and processing capability
- Performance and bound analysis
- Limited samples
- Adaptivity in dynamic situations

Joint domain design

- Fast- and slow-time waveforms
- Sparse array and sparse waveforms

Other considerations

- Strong interference
- Signal coherency

References

- S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Processing*, vol. 63, no. 6, pp. 1377-1390, March 2015. (2018 IEEE SPS Young Author Best Paper Award)
- S. Qin, Y. D. Zhang, M. G. Amin, and B. Himed, "DOA estimation exploiting a uniform linear array with multiple co-prime frequencies," *Signal Processing*, vol. 130, pp. 37-46, Jan. 2017. (2021 EURASIP Best Paper Award for Signal Processing)
- A. Ahmed, Y. D. Zhang, and B. Himed, "Effective nested array design for fourth-order cumulant-based DOA estimation," *IEEE Radar Conference*, May 2017. (Student Paper Competition Award - Third Place)
- C. Zhou, Y. Gu, X. Fan, Z. Shi, G. Mao, and Y. D. Zhang, "Direction-of-arrival estimation for coprime array via virtual array interpolation," *IEEE Trans. Signal Processing*, vol. 66, no. 22, pp. 5956-5971, Nov. 2018. (2021 IEEE SPS Young Author Best Paper Award)
- C. Zhou, Y. Gu, Z. Shi, and Y. D. Zhang, "Off-grid direction-of-arrival estimation using coprime array interpolation," *IEEE Signal Processing Letters*, vol. 25, no. 11, pp. 1710-1714, Nov. 2018.
- Z. Zheng, W-Q. Wang, Y. Kong, and Y. D. Zhang, "MISC Array: A new sparse array design achieving increased degrees of freedom and reduced mutual coupling effect," *IEEE Trans. Signal Processing*, vol. 67, no. 7, pp. 1728-1741, April 2019.
- S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "DOA estimation exploiting interpolated multi-frequency sparse array," *IEEE Sensor Array and Multichannel Signal Processing Workshop*, June 2020. (Best Student Paper Award)
- A. Ahmed, D. Silage, and Y. D. Zhang, "High-resolution target sensing using multi-frequency sparse array," *IEEE Sensor Array and Multichannel Signal Processing Workshop*, June 2020.
- S. Sun and Y. D. Zhang, "4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach," *IEEE Selected Topics in Signal Processing*, vol. 15, no. 4, pp. 879-891, June 2021.

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References

- S. Liu, Z. Mao, Y. D. Zhang, and Y. Huang, "Rank minimization-based Toeplitz reconstruction for DoA estimation using coprime array," *IEEE Communications Letters*, vol. 25, no. 7, pp. 2265-2269, July 2021.
- A. Ahmed and Y. D. Zhang, "Generalized non-redundant sparse array designs," *IEEE Trans. Signal Processing*, vol. 69, pp. 4580-4594, Aug. 2021.
- S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "Enhanced DOA estimation exploiting multi-frequency sparse array," *IEEE Trans. Signal Processing*, vol. 69, pp. 5935-5946, Oct. 2021.
- H. Zheng, C. Zhou, Z. Shi, Y. Gu, Y. D. Zhang, "Coarray tensor direction-of-arrival estimation," *IEEE Trans. Signal Processing*, vol. 71, pp. 1128-1142, March 2023.
- L. Xu, S. Sun, K. V. Mishra, and Y. D. Zhang, "Automotive FMCW radar with difference co-chirps," *IEEE Trans. Aerospace and Electronic Systems*, vol. 59, no. 6, pp. 8145-8165, Dec. 2023.
- C. Zhou, Y. Gu, Y. D. Zhang, and Z. Shi, "Sparse array interpolation for direction-of-arrival estimation," in M. G. Amin (ed.), *Sparse Arrays for Radar, Sonar, and Communications*, Wiley-IEEE Press, 2024.
- Y. D. Zhang and S. Sun, "Identical partitioning of consecutive integer set," *IEEE Sensor Array and Multichannel Signal Processing Workshop*, July 2024.
- A. Namboothiri, M. W. Chowdhury, and Y. D. Zhang, "Non-overlapping identical division of integer set," *Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Oct. 2024.
- M. W. T. S. Chowdhury, Y. D. Zhang, and B. Himed, "Sparse radar waveform design exploiting consecutive integer set partitioning," *IEEE International Radar Conference*, Atlanta, GA, May 2025.

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SIGNAL PROCESSING IN THE LAND OF ART, CULTURE AND BEAUTY

September 2025 - Palermo, Italy

Sparse Arrays and Sparse Waveforms: Design, Processing, and Applications
- Part I -

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