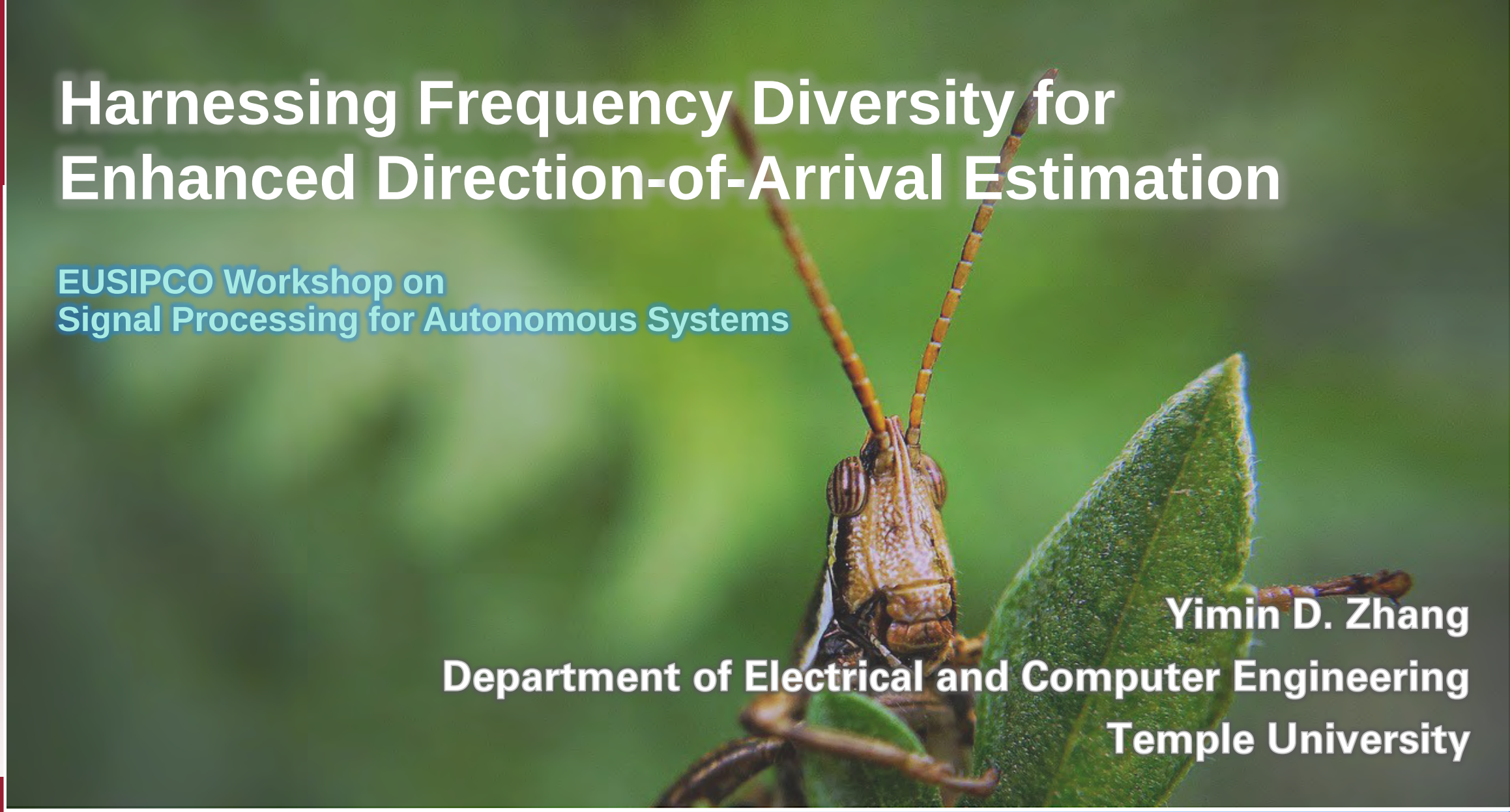
Harnessing Frequency Diversity for Enhanced Direction-of-Arrival Estimation

EUSIPCO Workshop on
Signal Processing for Autonomous Systems



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Outline

Harnessing Frequency Diversity for Enhanced Direction-of-Arrival Estimation

Section I. Sparse array in the context of difference coarray

Section II. Frequency diversity for DOA estimation

- **A. Coprime array exploiting multi-frequency signals**
- **B. Redundancy-free sparse arrays**
- **C. Multi-frequency vs frequency-switching**

Section III. Concluding Remarks

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Section I. Sparse array in the context of difference coarray

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A. Coprime array exploiting multi-frequency signals

B. Redundancy-free sparse arrays

C. Multi-frequency vs frequency-switching

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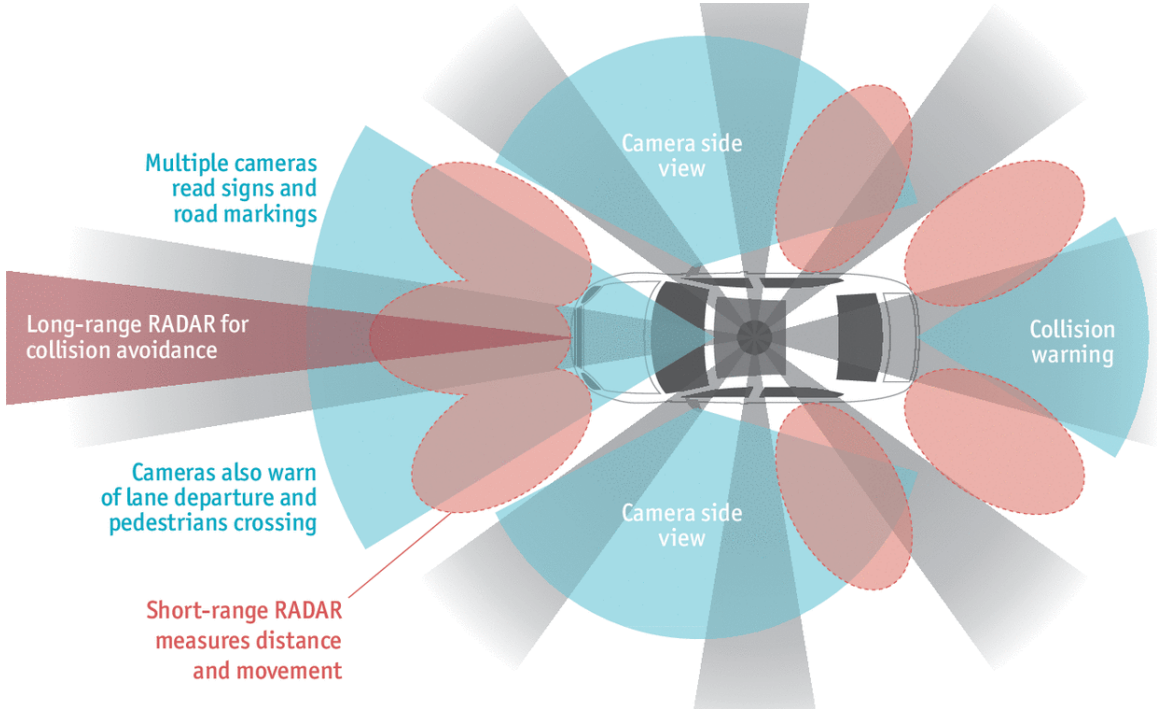
Applications of array processing

- Array processing uses multiple sensors (antennas, microphones, transducers) and plays a fundamental role in wireless communications, radar and sonar sensing, autonomous driving, speech separation, and medical imaging
- **Sensing: Localization/imaging**
  - Direction-of-arrival (DOA) estimation
  - Imaging

Applied to many autonomous systems such as automotive radar, as well as ground-to-air radar, sonar, and ultrasonic imaging

Significant wok has been done to utilize sparse arrays for DOA estimation.

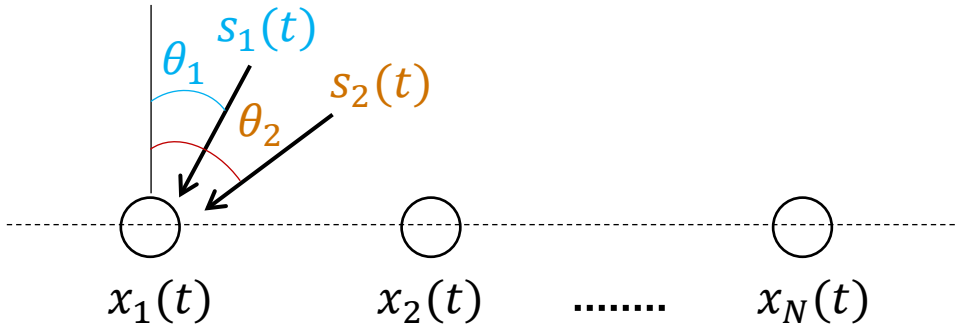
This talk will focus on how spectral resources can be harnessed to enhance the capability and performance for DOA estimation.



Signal model for ULA

Array signal model in the presence of K signals

$$\boldsymbol{x}(t) = \sum_{k=1}^{K} s_k(t)\boldsymbol{a}(\theta_k) + \boldsymbol{n}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t)$$



where

$\boldsymbol{a}(\theta_k)$: steering vector

$\mathbf{A} = [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_K), \cdots, \boldsymbol{a}(\theta_K)]$: array manifold matrix

$\boldsymbol{s}(t) = [s_1(t), \cdots, s_K(t)]^{\mathrm{T}}$: signal vector

We mainly consider linear arrays, but the results can be easily extended to planar arrays.

Covariance matrix: $\boldsymbol{R}_{xx} = \mathrm{E}[\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)] = \begin{bmatrix} \mathrm{E}[x_1(t)x_1^*(t)] & \mathrm{E}[x_1(t)x_2^*(t)] & \cdots & \mathrm{E}[x_1(t)x_N^*(t)] \\ \mathrm{E}[x_2(t)x_1^*(t)] & \mathrm{E}[x_2(t)x_2^*(t)] & \cdots & \mathrm{E}[x_2(t)x_N^*(t)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[x_N(t)x_1^*(t)] & \mathrm{E}[x_N(t)x_2^*(t)] & \cdots & \mathrm{E}[x_4(t)x_N^*(t)] \end{bmatrix}$

Estimated covariance matrix using T snapshots: $\widehat{\boldsymbol{R}}_{xx} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)$

Difference coarray

Subspace-based DOA estimation exploits the data covariance matrix $\boldsymbol{R}_{xx}$.

When the array is ULA, and the signals are uncorrected:

- $\boldsymbol{R}_{xx}$ is Toeplitz (diagonal-constant) and Hermitian
- $\boldsymbol{R}_{xx}$ is highly redundant: Only N elements are unique in the $N \times N$ covariance matrix
- We may not need N sensors to estimate the $N \times N$ covariance matrix

Consider removing the third sensor from a 4-element ULA:

- All the entries of the ULA covariance matrix can be recovered: e.g., $\mathrm{E}[x_2 x_3^*] \Rightarrow \mathrm{E}[x_1 x_2^*]$.
- The 4-element ULA and the 3-element sparse array are **different coarray equivalent** because they generate the same number of correlation lags.
- For physical array $\mathbb{G}$, the difference lags are given as: $\mathbb{C}_G = \{\boldsymbol{z} | \boldsymbol{z} = \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{G}\}$.

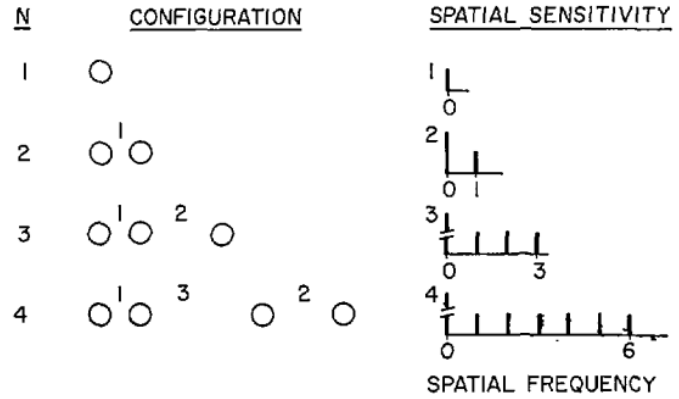
$$\boldsymbol{R}_{xx} = \begin{bmatrix} \mathrm{E}[x_1 x_1^*] & \mathrm{E}[x_1 x_2^*] & \mathrm{E}[x_1 x_3^*] & \mathrm{E}[x_1 x_4^*] \\ \mathrm{E}[x_2 x_1^*] & \mathrm{E}[x_2 x_2^*] & \mathrm{E}[x_2 x_3^*] & \mathrm{E}[x_2 x_4^*] \\ \mathrm{E}[x_3 x_1^*] & \mathrm{E}[x_3 x_2^*] & \mathrm{E}[x_3 x_3^*] & \mathrm{E}[x_3 x_4^*] \\ \mathrm{E}[x_4 x_1^*] & \mathrm{E}[x_4 x_2^*] & \mathrm{E}[x_4 x_3^*] & \mathrm{E}[x_4 x_4^*] \end{bmatrix}$$

Correlation lags (difference coarray)

-3 -2 -1 0 1 2 3

Sparse array design in the context of difference coarray

An attractive research direction of sparse array design is to have a desirable difference coarray, such as consecutive difference coarrays:

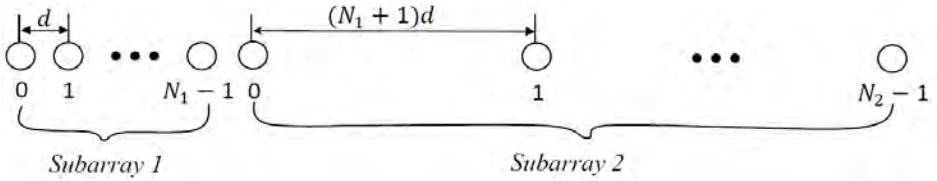


Non-systematical design

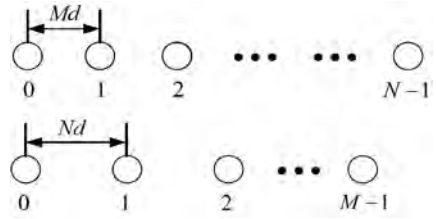
Minimum redundancy array (MRA) [1]: For a given number of physical sensors, MRA maximizes the number of consecutive virtual sensors in the resulting difference coarray: No systematical design approach.

Systematical design

Nested array [2]**:** provides consecutive lags from two uniform linear subarrays, one of which has a unit spacing; has high mutual coupling.



Coprime array [3]**:** utilizes a pair of uniform linear subarrays with M and N being coprime integers; Lags are generally not consecutive.



[1] A. Moffet, "Minimum-redundancy linear arrays," *IEEE Trans. Antennas and Propagation*, 1968.

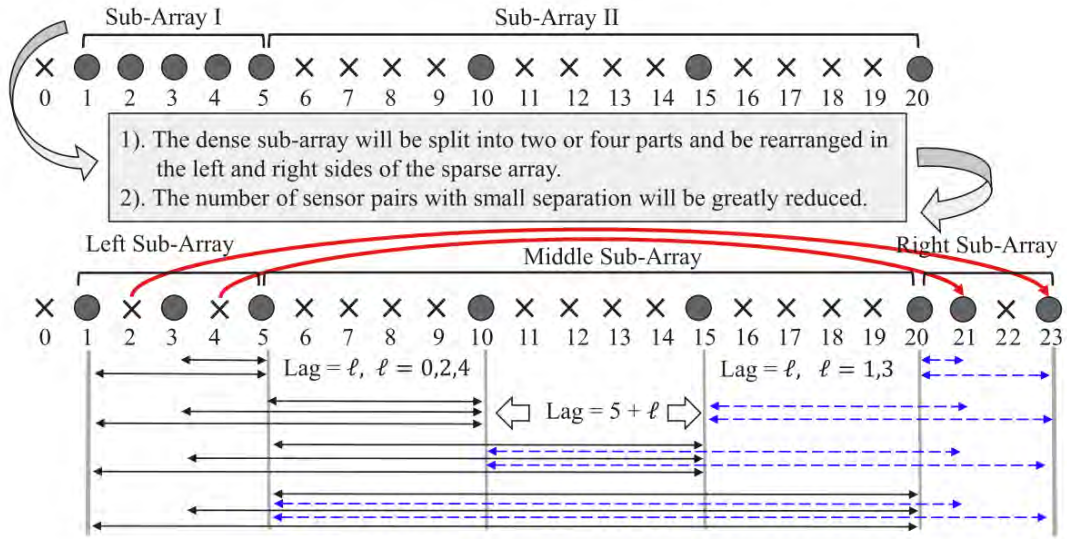
[2] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Processing*, 2010.

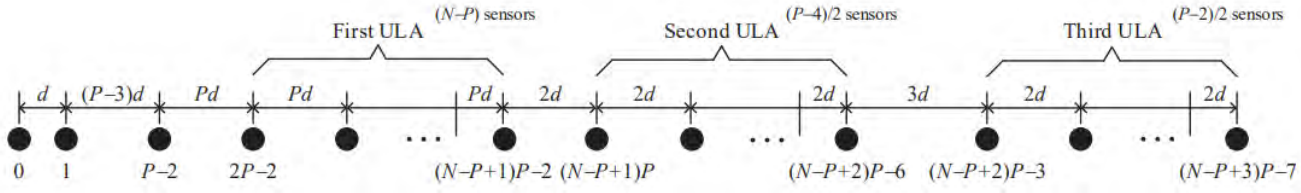
[3] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime sampler and arrays," *IEEE Trans. Signal Processing,* 2011.

More consecutive lags and less mutual coupling

Many sparse arrays are proposed for (i) more consecutive lags and (ii) less mutual coupling.

- **Augmented nested array** [4]**:** Split the densely located elements in inner subarray to reduce the mutual coupling. Several variations.
- **Thinned Coprime Array** [5]**:** provides the same number of consecutive lags, unique lags, and aperture as the conventional coprime array but with fewer sensors.
- **MISC** (maximum interelement spacing constraint) [6]: A four-segment configuration to achieve a high number of consecutive lags with low mutual coupling





[4] J. Liu, Y. Zhang, Y. Lu, S. Ren and S. Cao, "Augmented nested arrays with enhanced DOF and reduced mutual coupling," *IEEE Trans. Signal Processing*, 2017.

[5] A. Raza, W. Liu and Q. Shen, "Thinned coprime array for second-order difference co-array generation with reduced mutual coupling," *IEEE Trans. Signal Processing*, 2019.

[6] Z. Zheng, W-Q. Wang, Y. Kong, and Y. D. Zhang, "MISC Array: A new sparse array design achieving increased degrees of freedom and reduced mutual coupling effect," *IEEE Trans. Signal Processing*, 2019.

More consecutive lags and less mutual coupling: MISC family

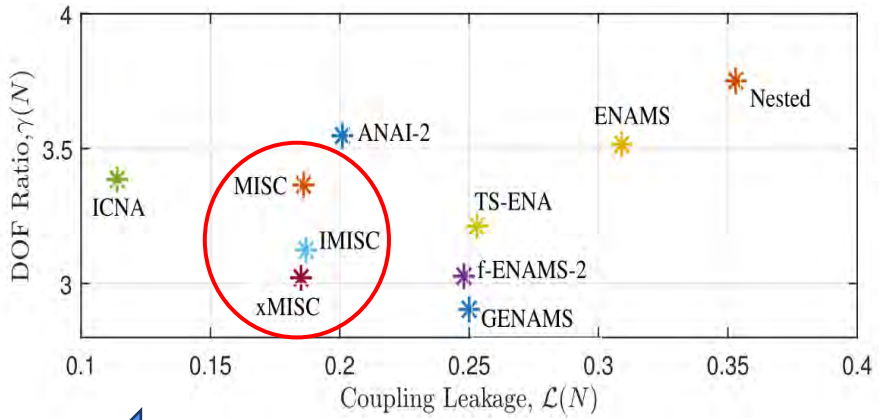
Many sparse arrays are proposed for (i) more consecutive lags and (ii) less mutual coupling.

MISC (maximum interelement spacing constraint) [6]: A four-segment configuration to achieve a high number of consecutive lags with low mutual coupling.

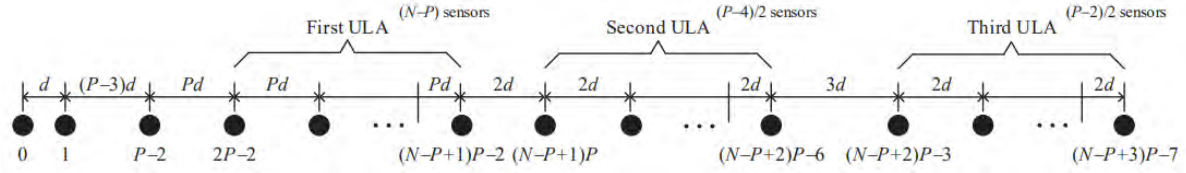
Modified versions: Use more segments to achieve higher freedom and better performance

- Improved MISC (I-MISC) [7]
- Extended MISC (xMISC) [8]

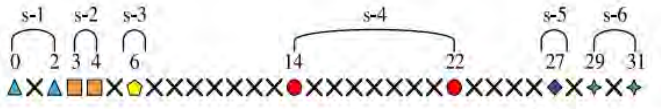
More degrees-of-freedom (DOFs)



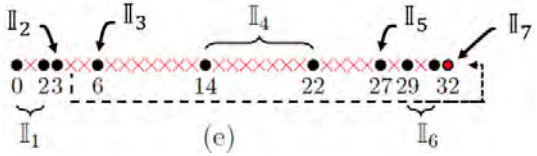
Lower mutual coupling effects



MISC (4 segments)



I-MISC (6 segments)



xMISC (7 segments)

[6] Z. Zheng, W-Q. Wang, Y. Kong, and Y. D. Zhang, "MISC Array: A new sparse array design achieving increased degrees of freedom and reduced mutual coupling effect," *IEEE Trans. Signal Processing*, 2019.

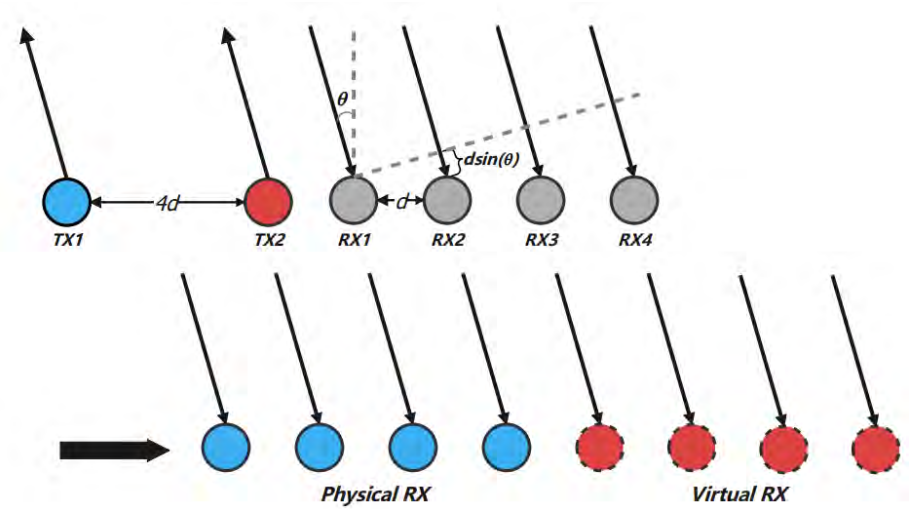
[7] W. Shi, Y. Li, and R. C. de Lamare, "Novel sparse array design based on the maximum inter-element spacing criterion," *IEEE Signal Processing Letters*, 2022.

[8] S. Wandale and K. Ichige, "xMISC: Improved sparse linear array via maximum inter-element spacing concept," *IEEE Signal Processing Letters*, 2023.

Additional approaches

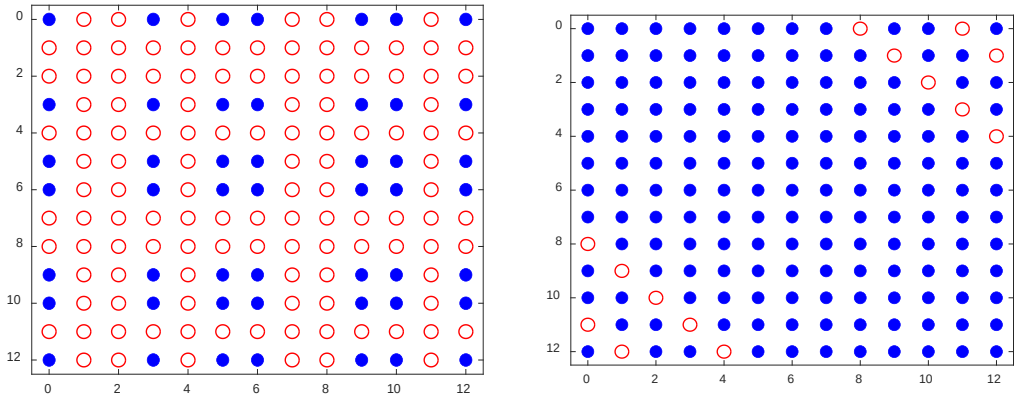
MIMO radar [9]

- The sum coarray in a MIMO radar is synthesized as $S = \{x + y \,|x \in \mathbb{S}_T\,, y \in \mathbb{S}_R\}$, where $\mathbb{S}_T$ and $\mathbb{S}_R$ are Tx and Rx antenna positions.
- Sparse MIMO radar achieves MN sum coarray entries using M Tx and N Rx antennas.



Sensor Interpolation [10]

- For sparse arrays, the covariance matrix of the corresponding ULA may be completed using such low-rank property.
- Because a matrix cannot be completed when an entire row or column is missing, it is important to utilize the Toeplitz and Hermitian structure of uniform arrays



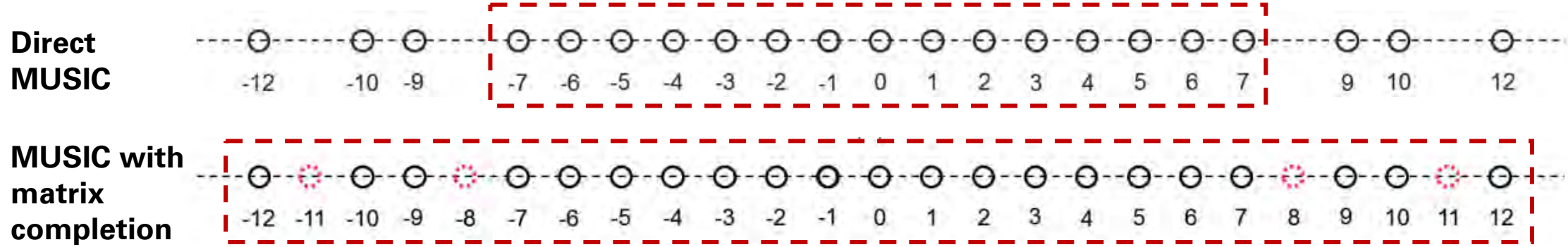
[9] J. Li and P. Stoica (eds.), *MIMO Radar Signal Processing*, Wiley, 2008.

[10] C. Zhou, Y. Gu, Z. Shi, and Y. D. Zhang, "Off-grid direction-of-arrival estimation using coprime array interpolation," *IEEE Signal Processing Letters*, 2018.

Matrix completion-aware sparse array design

Matrix completion

- Fills in information in missing lags
- Converts missing holes in the lag from obstacles in consecutive-lag construction into a resource for aperture extension
- Enabling off-grid DOA estimation with larger array apertures



With such capability, how shall we consider the “optimality” of a sparse array?

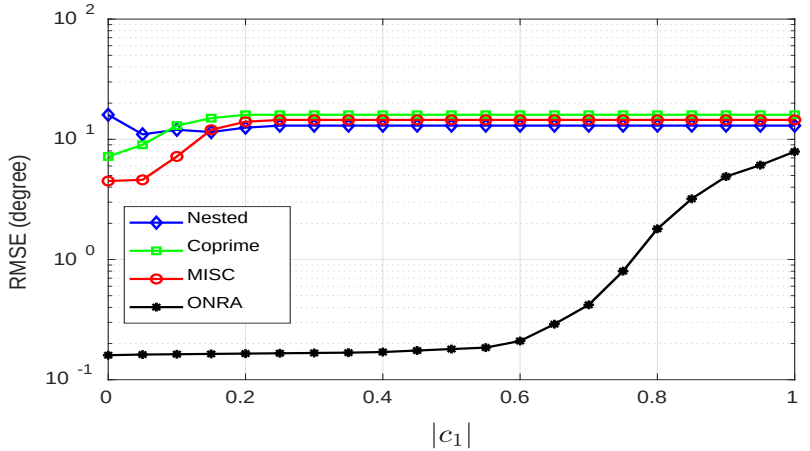
We introduce **optimized non-redundant array (ONRA)** [11]:

- Redundancy-free: Each lag only appears once (except lag-0)
- Introduce holes in the lag for reducing mutual coupling and enlarging array aperture
- Optimized using mixed-integer linear programming approach (not systematical)

[11] A. Ahmed and Y. D. Zhang, "Generalized non-redundant sparse array designs," *IEEE Trans. Signal Processing*, 2021.

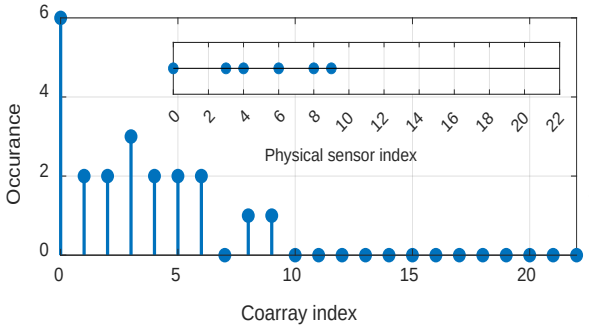
Non-redundant sparse array: Comparison

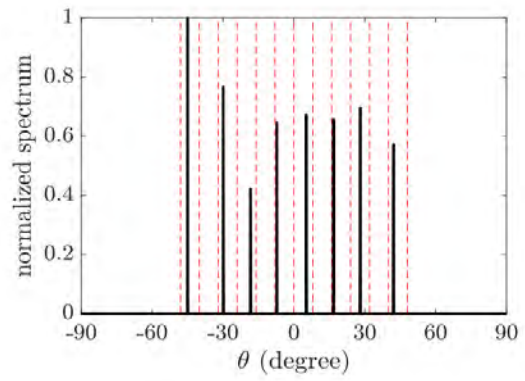
- Comparison for 6-sensor arrays (DOA estimation for 13 sources; LASSO) [11]
- ONRA has very low mutual coupling effect as the minimum interelement spacing is 2 units



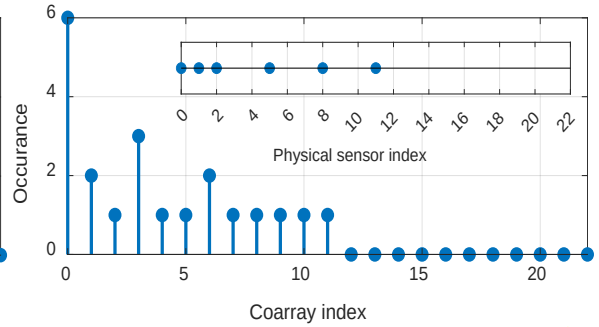
RMSE for two closely spaced source case

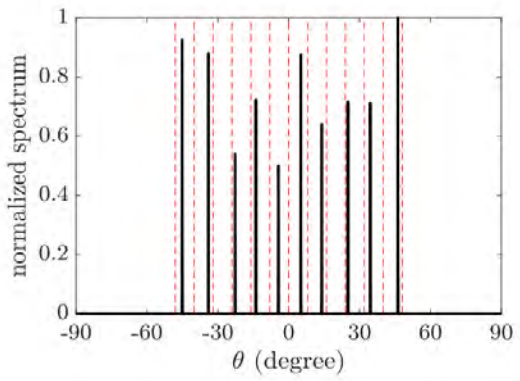
Coprime array (9 lags; max lag 9)



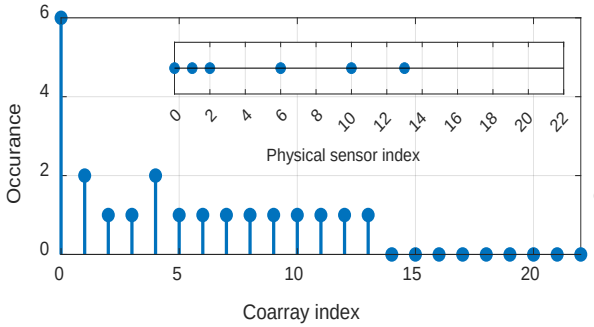


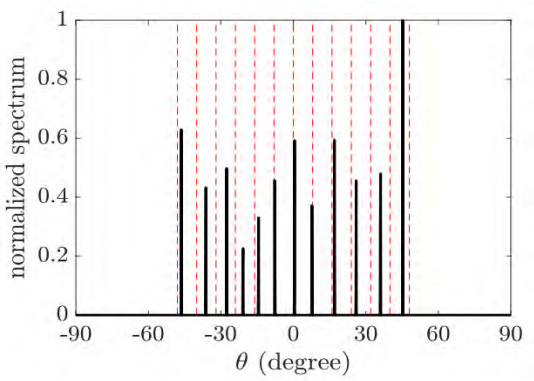
Nested array (12 lags; max lag 11)



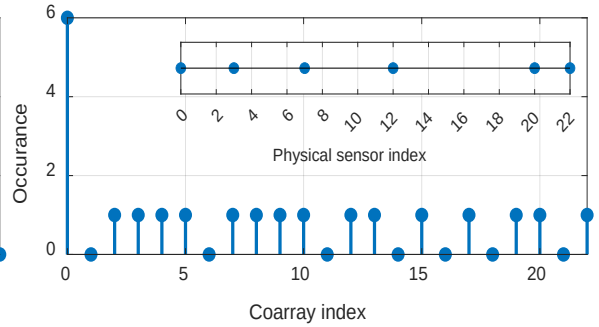


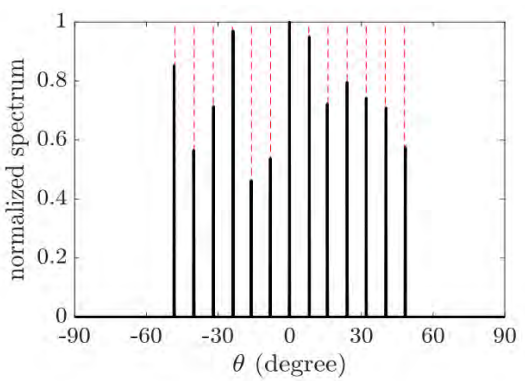
MISC array (14 lags; max lag 13)





ONRA (16 lags; max lag 22)





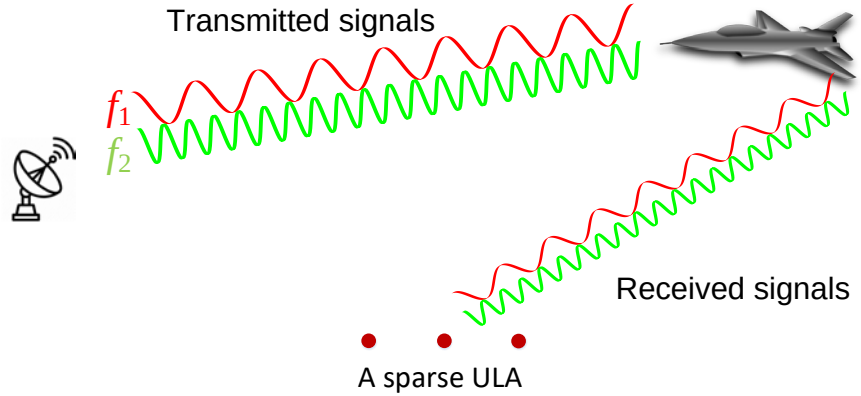
[11] A. Ahmed and Y. D. Zhang, "Generalized non-redundant sparse array designs," *IEEE Trans. Signal Processing*, 2021.

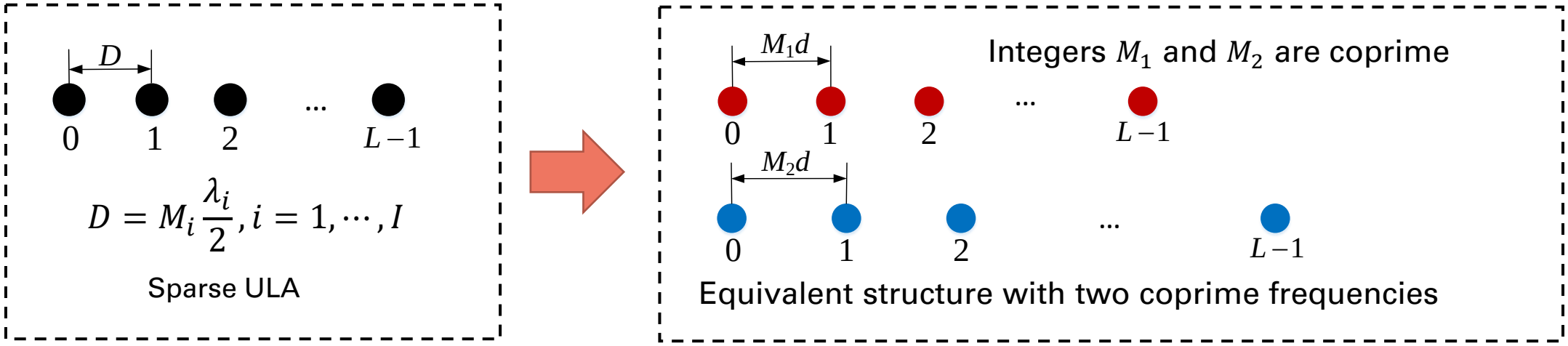
Outline

Harnessing Frequency Diversity for Enhanced Direction-of-Arrival Estimation

Multi-frequency sparse array

- Difference coarray is derived from the array data covariance matrix, **effectively trading time resources for additional spatial capacity**.
- Can we also **exploit frequency-domain resources to gain additional spatial capacity**?
- We start with considering a ULA with two frequencies that have a coprime relationship [12, 13].





[12] Y. D. Zhang, M. G. Amin, F. Ahmad, and B. Himed, "DOA estimation using a sparse uniform linear array with two CW signals of co-prime frequencies," *IEEE CAMSAP Workshop*, 2013.

[13] S. Qin, Y. D. Zhang, M. G. Amin, and B. Himed, "DOA estimation exploiting a uniform linear array with multiple co-prime frequencies," *Signal Processing*, 2017.

Signal model

Assume I continuous-wave (CW) signals with coprime frequencies.

For frequency f_i, RF signal received from K far-field targets at L-element ULA with spacing D:

$$\tilde{\boldsymbol{x}}_i(t) = \exp(j2\pi f_i t)\sum_{k=1}^{K}\rho_k^{(i)}(t)\boldsymbol{a}_i(\theta_k) + \tilde{\boldsymbol{n}}_i(t), i = 1, 2$$

where

$\rho_k^{(i)}(t)$: complex envelop of uncorrelated signal from target k corresponding to f_i

$\boldsymbol{a}_i(\theta_k) = \left[1, e^{-j\frac{2\pi D}{\lambda_i}\sin(\theta_k)}, \cdots, e^{-j\frac{2\pi D}{\lambda_i}\sin(\theta_k)}\right]^T$: steering vector for target k corresponding to f_i

It is important to note that $\rho_k^{(i)}(t)$ **is in general frequency-dependent** due to the different propagation phase delays.

Baseband signal after down-conversion:

$$\boldsymbol{x}_i(t) = \sum_{k=1}^{K}\rho_k^{(i)}(t)\boldsymbol{a}_i(\theta_k) + \boldsymbol{n}_i(t) = \boldsymbol{A}_i\boldsymbol{s}_i(t) + \boldsymbol{n}_i(t)$$

where

$$\boldsymbol{A}_i = [a_i(\theta_1), \cdots, \boldsymbol{a}_i(\theta_K)],\ \ s_i(t) = \left[\rho_1^{(i)}(t), \cdots, \rho_K^{(i)}(t)\right]^T$$

Signal model

Let $D = M_i\lambda_i/2$ (or $M_i = 2D/\lambda_i$) to be integer multiples of half-wavelength at all frequencies with $M_1 < M_2$, then :

$$\boldsymbol{a}_i(\theta_k) = \left[1, e^{-jM_i\pi \sin(\theta_k)}, \cdots, e^{-jM_i(L-1)\pi \sin(\theta_k)}\right]^T$$

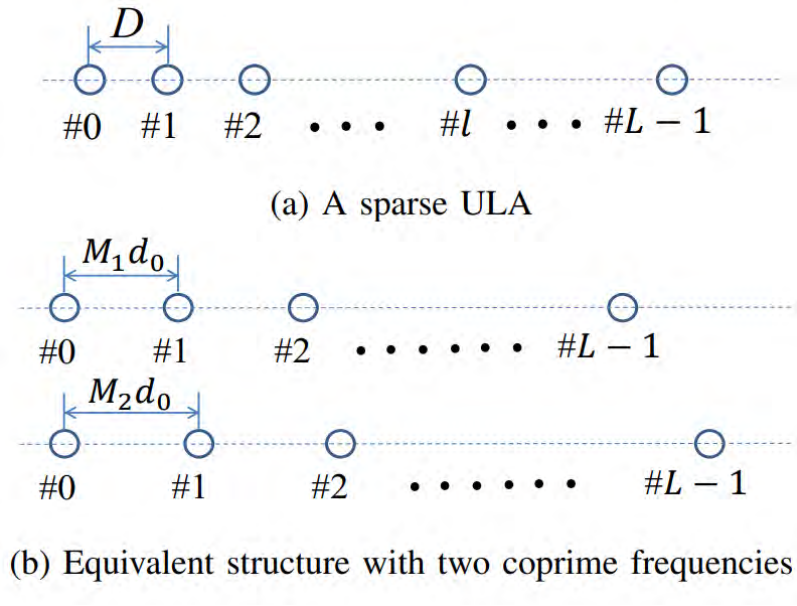
As such, we obtain virtual antennas at

$$\mathbb{P} = \{M_1 l_1 d_0 | 0 \le l_1 \le L-1\} \cup \{M_2 l_2 d_0 | 0 \le l_2 \le L-1\}$$

where

d_0: half-wavelength unit in a normalized frequency

l_1 and l_2: indexes of sensor positions



(a) A sparse ULA

(b) Equivalent structure with two coprime frequencies

Important observations:

- A virtual coprime array with two uniform subarrays are obtained from a single physical ULA.
- Each subarray has the same number of L sensors.
- DOA estimation needs to account for the fact that signals corresponding to different virtual arrays have distinct phases.
- Group sparse compressive sensing (CS)-based methods are effective in this case.

Proportional spectra scenario

With I frequencies and L antennas, the maximum number of achievable unique lags: [13]

$$\eta = (L^2-1)(I^2-I) - 2(L-1)(I^2-2I) + 1 \propto O(I^2L^2)$$

- The actual number of unique lags is reduced by redundant lags
- High number of lags requires a large separation between different multipliers $M_i, i = 1,\ldots,I$

Proportional spectra scenario: [14] $\rho_k^{(i)}(t)$ does not vary with frequency

- For two-frequency case, $\boldsymbol{s}_1(t) = \boldsymbol{s}_2(t) = \boldsymbol{s}(t)$.
- Stacking $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ renders

$$\boldsymbol{x}(t) = \begin{bmatrix}\boldsymbol{x}_1(t)\\ \boldsymbol{x}_2(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_1\boldsymbol{s}_1(t)\\ \boldsymbol{A}_2\boldsymbol{s}_2(t)\end{bmatrix} + \boldsymbol{n}(t) = \begin{bmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_2\end{bmatrix}\boldsymbol{s}(t) + \boldsymbol{n}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t)$$

where $\boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_2\end{bmatrix}$ with $\boldsymbol{a}(\theta_k) = \begin{bmatrix}\boldsymbol{a}_1(\theta_k)\\ \boldsymbol{a}_2(\theta_k)\end{bmatrix}$

In this case, the two subarrays obtained from different frequencies render an augmented array, and existing DOA estimation for sparse arrays can be used.

[13] S. Qin, Y. D. Zhang, M. G. Amin, and B. Himed, "DOA estimation exploiting a uniform linear array with multiple co-prime frequencies," *Signal Processing*, 2017.

[14] E. BouDaher, Y. Jia, F. Ahmad, and M. G. Amin, "Multifrequency co-prime arrays for high-resolution direction-of-arrival estimation," *IEEE Trans. Signal Processing*, 2015.

General case with unproportional spectra

- **General case:** $\rho_k^{(i)}(t) \neq \rho_k^{(j)}(t)$ while $\left|\rho_k^{(i)}(t)\right| = \left|\rho_k^{(j)}(t)\right| = 1$
- Received signal vector for frequency i from K sources

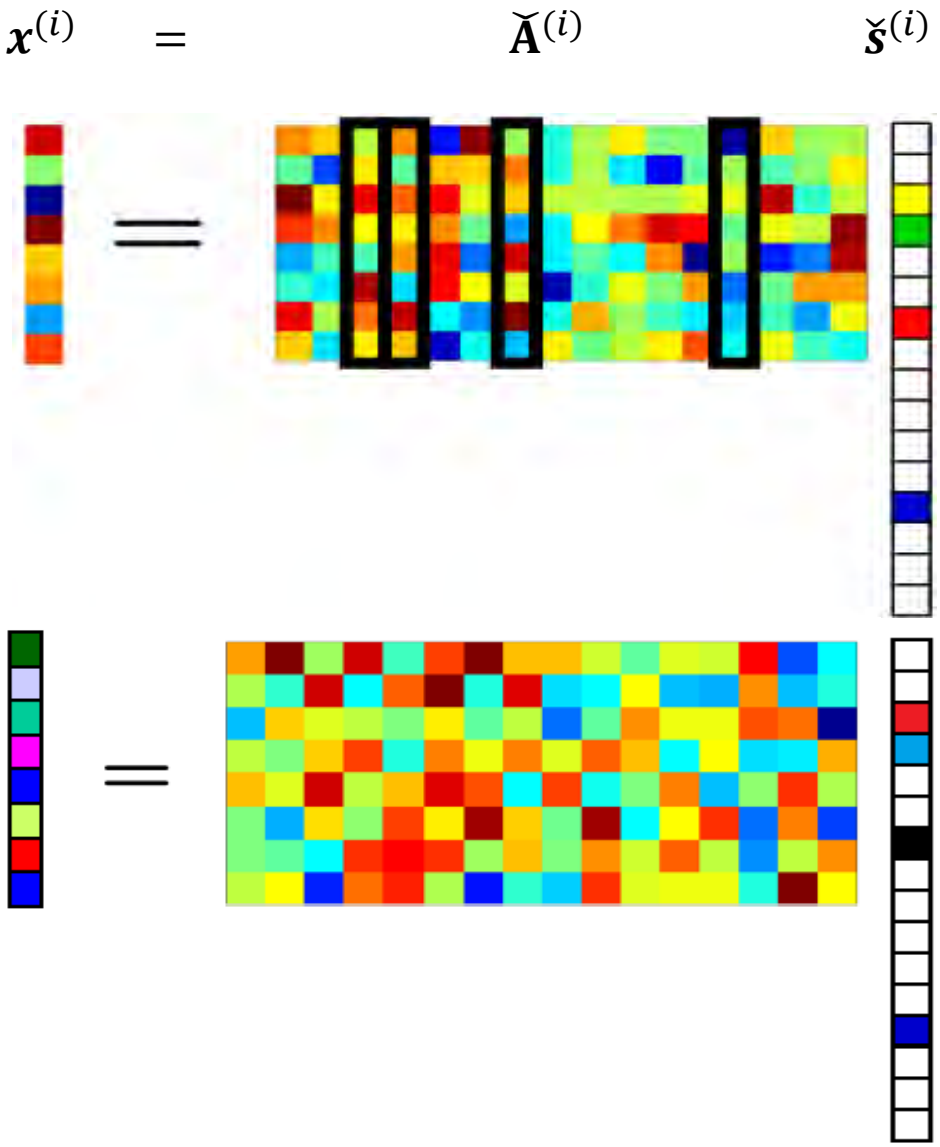
$$\boldsymbol{x}_i(t) = \sum_{k=1}^{K} \rho_k^{(i)}(t)\boldsymbol{a}_i(\theta_k) + \boldsymbol{n}_i(t) = \boldsymbol{A}_i\boldsymbol{s}_i(t) + \boldsymbol{n}_i(t)$$

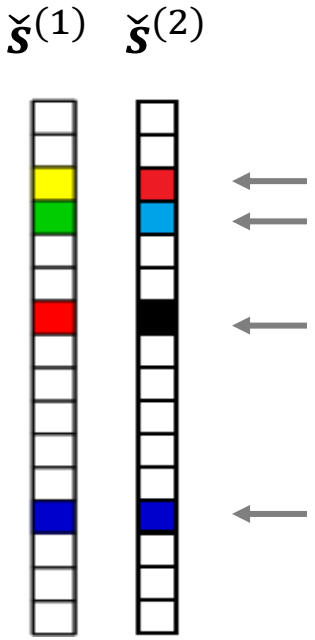
- We obtain I^2 covariance matrices

$$\boldsymbol{R}_{\boldsymbol{x}_i\boldsymbol{x}_j} = \mathrm{E}\{\boldsymbol{x}_i(t)\boldsymbol{x}_j^H(t)\} = \begin{cases} \boldsymbol{A}_i\boldsymbol{R}_{ii}\boldsymbol{A}_i^H + \sigma_n^2\boldsymbol{I}_L, & i = j \\ \boldsymbol{A}_i\boldsymbol{R}_{ij}\boldsymbol{A}_j^H, & i \neq j \end{cases}$$

- Key observations:
  - $\boldsymbol{R}_{ii}$ is real diagonal, but $\boldsymbol{R}_{ij}$ is generally complex and differs for different frequency pairs
  - We can choose to use only the cross-lags which retain all the lags. Standard CS methods can be used.
  - To utilize both auto- and cross-lags, group sparsity is used to account for the different values of the signal phases.

Group sparsity





Same positions of nonzero values, but generally with different values

Group sparsity

- Group Lasso [15] and block OMP [16] use mixed l_2/l_1-norm (l_2-norm of the absolute values; also called mixed l_{12}-norm) to handle signal group sparsity.
- l_1-norm relaxation in OMP/Lasso
  ➔ mixed $l_{1,2}$-norm relaxation in group sparsity version
- Group Lasso:

$$\min \quad \|\boldsymbol{x}\|_{1,2} \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2 \leq \epsilon$$

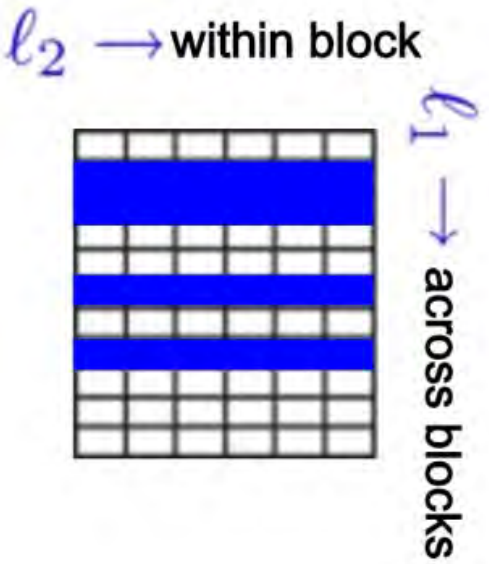
where

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}^{(1)} \\ \boldsymbol{y}^{(2)} \\ \vdots \\ \boldsymbol{y}^{(L)} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{(1)} \\ \boldsymbol{x}^{(2)} \\ \vdots \\ \boldsymbol{x}^{(L)} \end{bmatrix} \qquad \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}^{(1)} & & & \\ & \boldsymbol{\Phi}^{(2)} & & \\ & & \ddots & \\ & & & \boldsymbol{\Phi}^{(L)} \end{bmatrix}$$

or (multi-measurement vector)

$$\min \quad \|\boldsymbol{X}\|_{1,2} \quad \text{s.t.} \quad \|\boldsymbol{Y} - \boldsymbol{\Phi}\boldsymbol{X}\|_F \leq \epsilon$$

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{y}^{(1)} & \boldsymbol{y}^{(2)} \cdots \boldsymbol{y}^{(L)} \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}^{(1)} & \boldsymbol{x}^{(2)} \cdots \boldsymbol{x}^{(L)} \end{bmatrix}$$



[15] L. Jacob, G. Obozinski, and J-P. Vert, "Group Lasso with overlap and graph Lasso," *International Conference on Machine Learning*, 2009.

[16] Y. C. Eldar, P. Kuppinger, and H. Bölcskei, "Block-sparse signals: Uncertainty relations and efficient recovery," *IEEE Trans. Signal Processing*, 2010.

Unequal coefficient case

Auto-lag and cross-lag covariance matrices

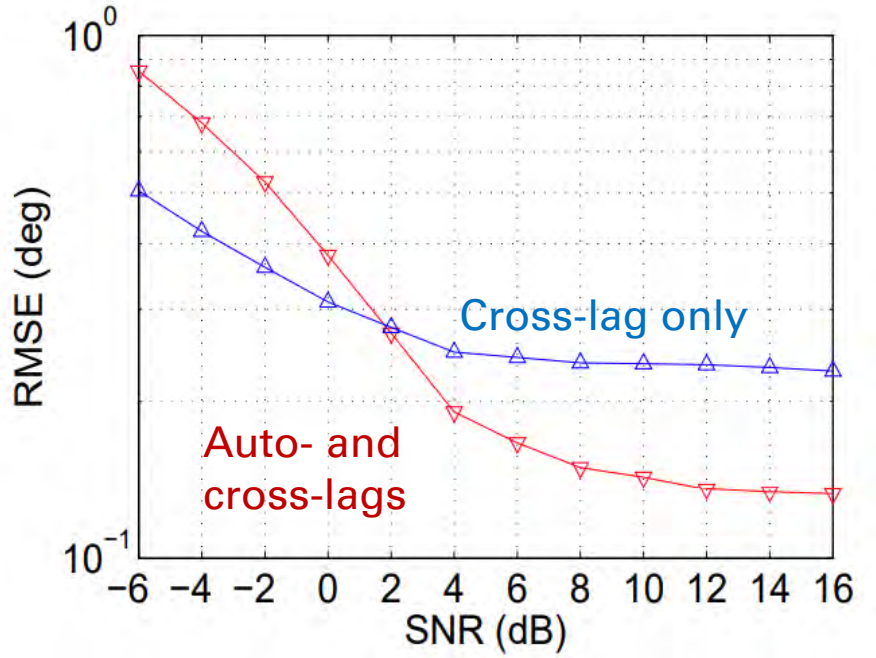
$$\boldsymbol{R}_{\boldsymbol{x}_i\boldsymbol{x}_j} = \boldsymbol{A}_i\boldsymbol{R}_{ii}\boldsymbol{A}_i^H + \sigma_n^2\boldsymbol{I}_L$$

$$\boldsymbol{R}_{\boldsymbol{x}_i\boldsymbol{x}_j} = \boldsymbol{A}_i\boldsymbol{R}_{ij}\boldsymbol{A}_j^H, i \neq j$$

Vectorized group sparsity problem

$$\boldsymbol{z}_{ij} = \widetilde{\boldsymbol{B}}_{ij}^o\widetilde{\boldsymbol{b}}_{ij}^o + \boldsymbol{\epsilon}_{ij}$$

$$\widetilde{\boldsymbol{B}}_{ij}^o = \begin{cases} [\boldsymbol{A}_{ii}^o, \boldsymbol{i}] \\ [\boldsymbol{A}_{ij}^o, \boldsymbol{0}] \end{cases}$$



RMSE versus input SNR
($M_1 = 3$, $M_2 = 4$, $Q = 6$, and 2000 snapshots)

Example:

- At a moderate or high SNR, the utilization of both auto- and cross-lag covariances benefits from additional measurement offered by the auto-lags, resulting in the improved performance than the cross-lag only scenario.
- In the low SNR region, the performance of the algorithm using cross-lag covariances only is better. The inclusion of auto-lag covariance matrices causes additional errors in the noise power estimation, whereas this term does not exist in the cross-lag covariances.

[13] S. Qin, Y. D. Zhang, M. G. Amin, and B. Himed, "DOA estimation exploiting a uniform linear array with multiple co-prime frequencies," *Signal Processing*, 2017.

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Section I. Sparse array in the context of difference coarray

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C. Multi-frequency vs frequency-switching

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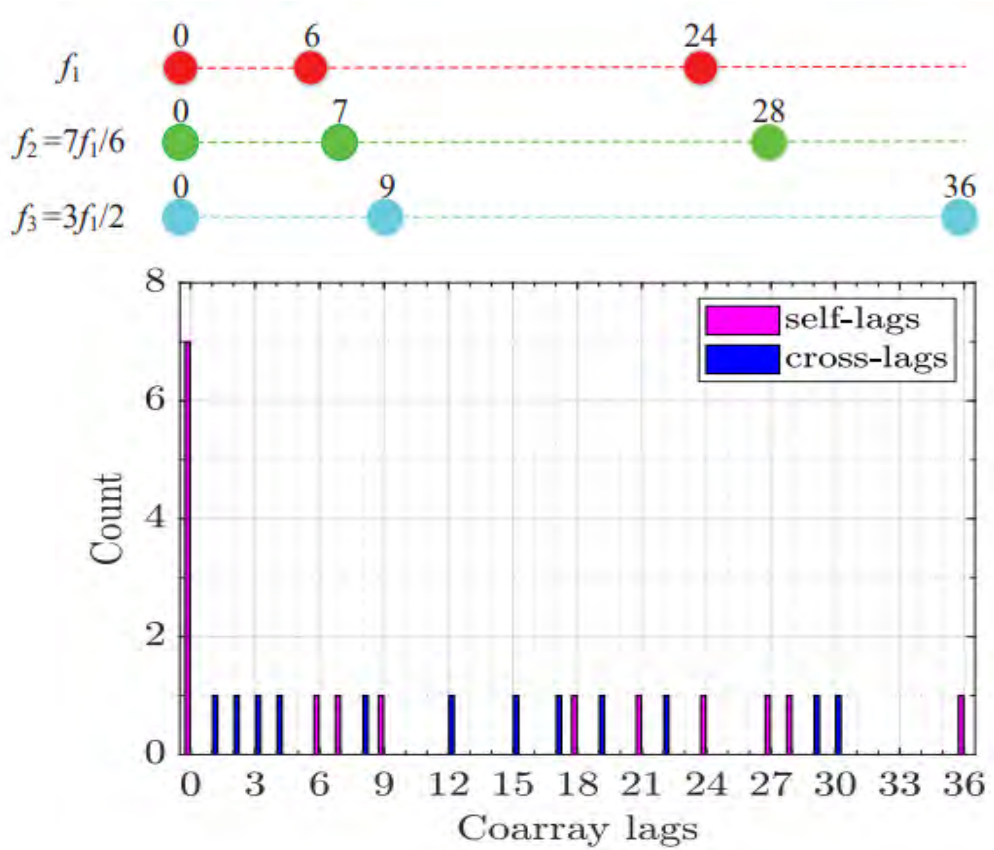
Multi-frequency sparse array

Using physical ULA for multi-frequency array processing results in a high number of lag redundancies.

Further enhancements are considered [17]:

- **Nonuniform physical array**: Reduce correlation lag redundancies to achieve a higher number of DOFs
- **Redundancy-free design:** By carefully chosen the sensor placement and frequencies, the virtual sensors corresponding to different frequencies do not overlap.
- **Modified array interpolation:** Enable information fusion from multi-frequency signals and achieve robust DOA estimation

Example: 3 antennas, 3 frequencies



7 virtual sensors; 10 non-negative self-lags; 12 non-negative cross-lags

[17] S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "Enhanced DOA estimation exploiting multi-frequency sparse array," *IEEE Trans. Signal Processing*, 2021.

Multi-frequency sparse array

- Received signal vector for frequency i from K sources

$$\boldsymbol{x}_{\mathbb{S}_i}(t) = \sum_{k=1}^{K} \rho_k^{(i)}(t)\boldsymbol{a}_{\mathbb{S}_i}(\theta_k) + \boldsymbol{n}_{\mathbb{S}_i}(t)$$

- We obtain I^2 covariance matrices

$$\boldsymbol{R}_{\boldsymbol{x}_{\mathbb{S}_i}\boldsymbol{x}_{\mathbb{S}_j}} = \mathrm{E}\left\{\boldsymbol{x}_{\mathbb{S}_i}(t)\boldsymbol{x}_{\mathbb{S}_j}^H(t)\right\} = \begin{cases} \boldsymbol{A}_{\mathbb{S}_i}\boldsymbol{R}_{\mathbb{S}_i\mathbb{S}_i}\boldsymbol{A}_{\mathbb{S}_i}^H + \sigma_n^2\boldsymbol{I}_L, i = j \\ \boldsymbol{A}_{\mathbb{S}_i}\boldsymbol{R}_{\mathbb{S}_i\mathbb{S}_j}\boldsymbol{A}_{\mathbb{S}_j}^H, \quad i \neq j \end{cases}$$

- Important to note that $\rho_k^{(i)}(t) \neq \rho_k^{(j)}(t)$
  - $\boldsymbol{R}_{\mathbb{S}_i\mathbb{S}_i}$ is real diagonal, but $\boldsymbol{R}_{\mathbb{S}_i\mathbb{S}_j}$ is generally complex and differs for different frequency pairs
  - The signals observed at different frequencies cannot be directly combined or stacked together
  - However, because they share the same signal DOAs, group sparsity-based approaches are effective to perform DOA estimation

Multi-frequency sparse array

Array interpolation using self-lags only

- Construct $\boldsymbol{R}^{(i)}_{\boldsymbol{x}_{\mathbb{U}}\boldsymbol{x}_{\mathbb{U}}}$ corresponding to the ULA for frequency i from $\boldsymbol{R}_{\boldsymbol{x}_{\mathbb{S}_i}\boldsymbol{x}_{\mathbb{S}_i}}$, and estimate $\boldsymbol{R}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}$ as

$$\boldsymbol{R}_{y_{\mathbb{U}}y_{\mathbb{U}}} = \left(\sum_{i=1}^{I} \boldsymbol{R}^{(i)}_{\boldsymbol{x}_{\mathbb{U}}\boldsymbol{x}_{\mathbb{U}}} \circ \boldsymbol{B}_i\right) \circ \boldsymbol{D}$$

where $B_i(m,n)$ denotes binary mask,

$$D(m,n) = \frac{1}{\sum_{i=1}^{I} B_i(m,n) + \epsilon}$$

and $\epsilon > 0$ is a small positive value

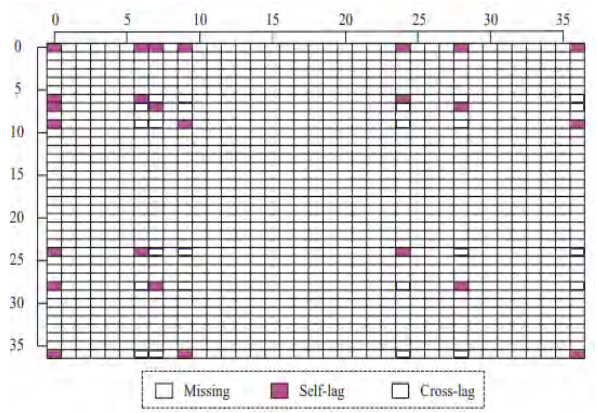
- Sparsity-based optimization

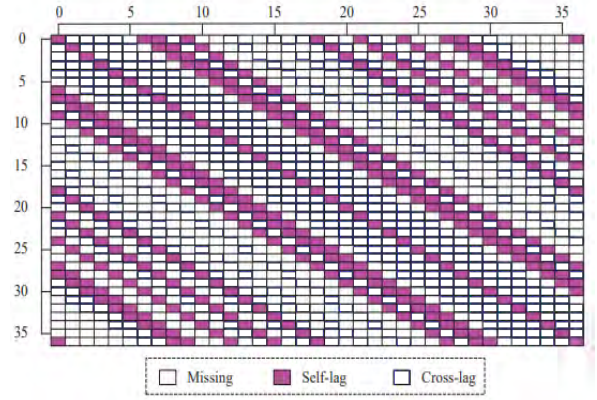
$$\min_{\boldsymbol{w},\boldsymbol{R}} \quad \left\|\boldsymbol{\mathcal{T}}(\boldsymbol{w}) \circ \boldsymbol{B} - \boldsymbol{R}_{y_{\mathbb{U}}y_{\mathbb{U}}}\right\|_F^2 + \zeta \ \mathrm{Tr}\ (\boldsymbol{\mathcal{T}}(\boldsymbol{w}))$$

$$\text{s.t.} \quad \boldsymbol{\mathcal{T}}(\boldsymbol{w}) \succcurlyeq 0$$

$\mathcal{T}(\boldsymbol{w})$: Hermitian Toeplitz matrix with $\boldsymbol{w}$ as the first column

ζ: regularization parameter





In a Hermitian Toeplitz matrix, a single column w uniquely specifies all the elements of the matrix.

$$\boldsymbol{\mathcal{T}}(\boldsymbol{w}) = \begin{bmatrix} \boldsymbol{w_1} & w_2^* & w_3^* & w_4^* \\ \boldsymbol{w_2} & w_1 & w_2^* & w_3^* \\ \boldsymbol{w_3} & w_2 & w_1 & w_2^* \\ \boldsymbol{w_4} & w_3 & w_2 & w_1 \end{bmatrix}$$

$\boldsymbol{w}$

Multi-frequency sparse array

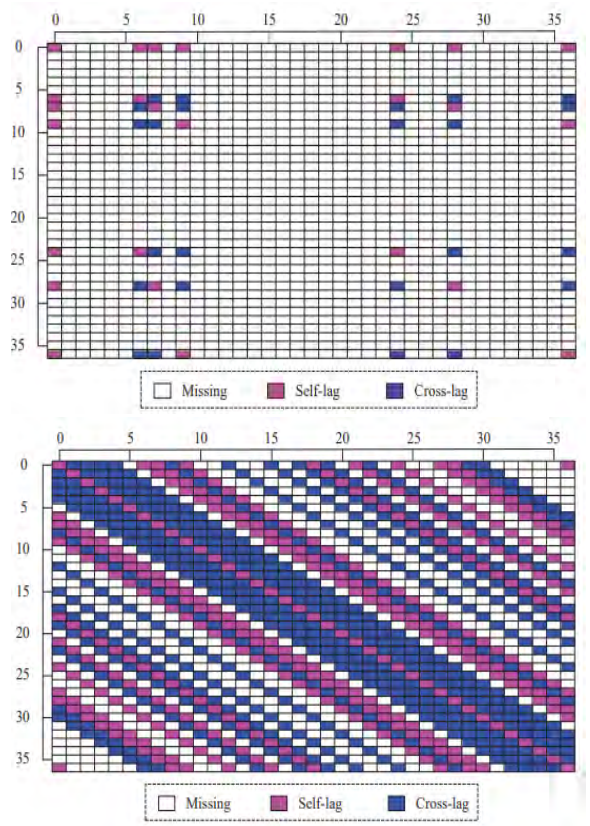
Array interpolation using both self- and cross-lags

- Exploiting both self- and cross-lags more effective utilization of the observed information.
- Vectorizing $\boldsymbol{R}_{\boldsymbol{x}_{\mathbb{S}_i}\boldsymbol{x}_{\mathbb{S}_j}}$, $i \neq j$, renders

$$\boldsymbol{z}_{\mathbb{S}_i\mathbb{S}_j} = \mathrm{vec}\left\{\boldsymbol{R}_{\boldsymbol{x}_{\mathbb{S}_i}\boldsymbol{x}_{\mathbb{S}_j}}\right\} = \widetilde{\boldsymbol{A}}_{\mathbb{S}_i\mathbb{S}_j}\boldsymbol{r}_{i,j} = \boldsymbol{\Phi}_{\mathbb{S}_i\mathbb{S}_j}\tilde{\boldsymbol{r}}_{i,j}$$

  where $\boldsymbol{\Phi}_{\mathbb{S}_i\mathbb{S}_j} = [\widetilde{\boldsymbol{A}}_{\mathbb{S}_i\mathbb{S}_j} \;\; 0]$ is the dictionary counting for noise term

- Because $\boldsymbol{r}_{i,j}$ differs to each other, we need to solve them for all frequency pairs, rendering a high number of unknowns
- However, all these unknown vectors should have the same **support**, i.e., same nonzero positions indicating the same signal DOAs.
- Such property is referred to as **group sparsity,** and group compressive sensing methods can achieve near-coherent data fusion.



Multi-frequency sparse array

Group sparse reconstruction

- Achieve near-coherent data fusion
- Useful in DOA estimation when involving unknown phase

Array Interpolation Using Both Self- and Cross-lags (con't)

- Vectorizing $\boldsymbol{\mathcal{T}}(\boldsymbol{w})$ renders $z_{\mathbb{U}\mathbb{U}} = \mathrm{vec}\big(\boldsymbol{\mathcal{T}}(\boldsymbol{w})\big) = \boldsymbol{\Phi}_{\mathbb{U}\mathbb{U}}\,\boldsymbol{r}$
- Group sparsity-based optimization

$$\min_{\boldsymbol{w},\,\boldsymbol{\mathcal{R}}} \left\| \boldsymbol{\mathcal{T}}(\boldsymbol{w}) \circ \boldsymbol{B} - \boldsymbol{R}_{y_{\mathbb{U}} y_{\mathbb{U}}} \right\|_{\mathrm{F}}^2 + \zeta\ \mathrm{Tr}\ \left(\boldsymbol{\mathcal{T}}(\boldsymbol{w})\right) + \beta_1 \sum_{1 \le i < j \le I} \left\| \boldsymbol{z}_{\mathbb{S}_i \mathbb{S}_j} - \boldsymbol{\Phi}_{\mathbb{S}_i \mathbb{S}_j}\, \boldsymbol{r}_{i,j} \right\|_2 + \beta_1 \| z_{\mathbb{U}\mathbb{U}} - \boldsymbol{\Phi}_{\mathbb{U}\mathbb{U}} \boldsymbol{r} \|_2 + \beta_2 \| \boldsymbol{\mathcal{R}} \|_{1,2}$$

$$\text{s.t. } \boldsymbol{\mathcal{T}}(\boldsymbol{w}) \succcurlyeq 0$$

where $\boldsymbol{\mathcal{R}} = [\boldsymbol{r}_{1,2}, \boldsymbol{r}_{1,3}, \ldots, \boldsymbol{r}_{I-1,I}, \boldsymbol{r}]$, β_1 and β_2 are regularization parameters, and the $l_{1,2}$-norm of $\boldsymbol{\mathcal{R}}$ is

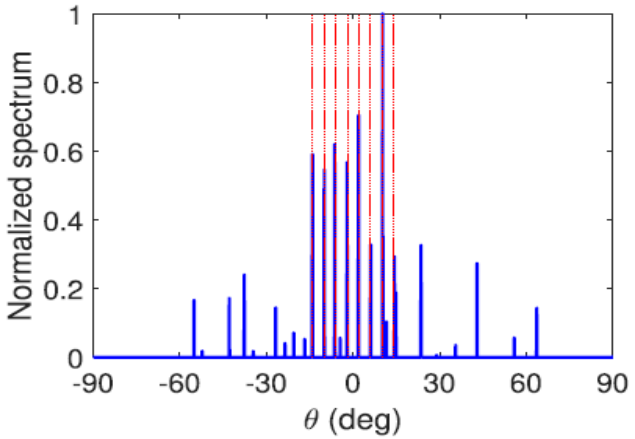
$$\| \boldsymbol{\mathcal{R}} \|_{1,2} = \sum_{m=1}^{G+1} \left(\sum_{n=1}^{\frac{I(I-1)}{2}+1} \boldsymbol{\mathcal{R}}(m,n) \boldsymbol{\mathcal{R}}^*(m,n) \right)^{1/2}$$

G: dimension of $\boldsymbol{r}$ (number of angular grid points)

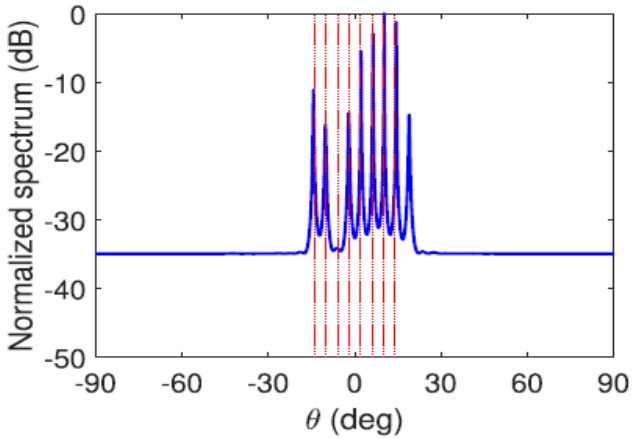
Multi-frequency sparse array

Simulation example:

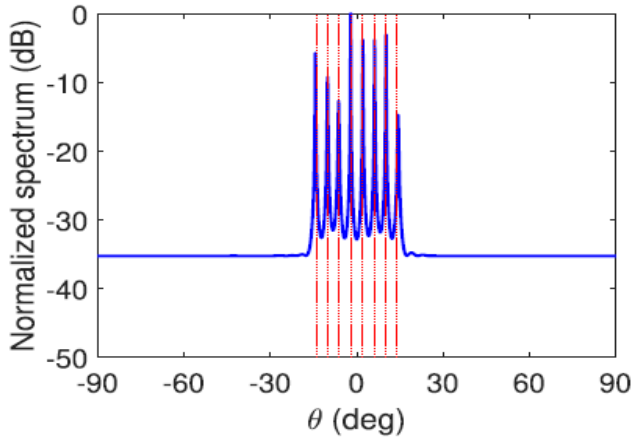
- $K = 8$ targets are uniformly distributed in [−14º, 14º]
- Input SNR = 10 dB for all targets
- $T = 100$ snapshots



Group lasso without array interpolation [Ahmed 2020]



MUSIC after array interpolation with self-lags [Zhang 2020]



MUSIC using both self- and cross-lags [Zhang 2021]

[18] A. Ahmed, D. Silage, and Y. D. Zhang, "High-resolution target sensing using multi-frequency sparse array," *IEEE SAM Workshop*, 2020.
[19] S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "DOA estimation exploiting interpolated multi-frequency sparse array," *IEEE SAM Workshop*, 2020.
[17] S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "Enhanced DOA estimation exploiting multi-frequency sparse array," *IEEE Trans. Signal Processing*, 2021.

Array interpolation

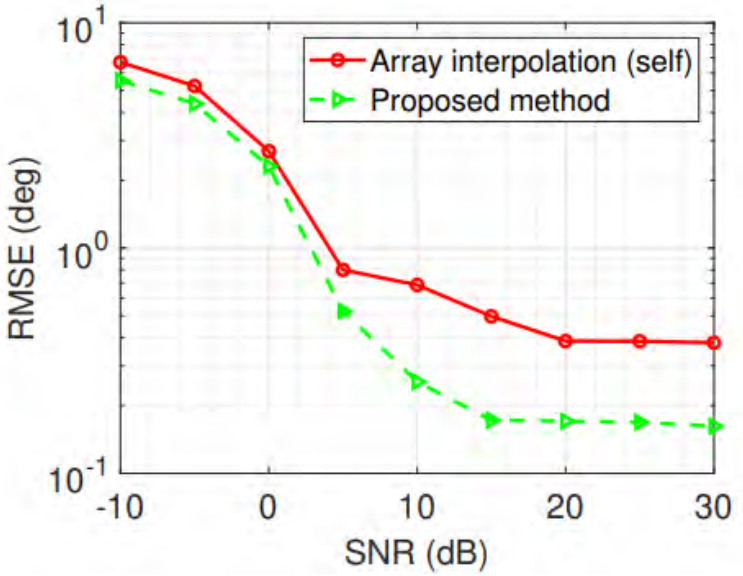
RMSE example:

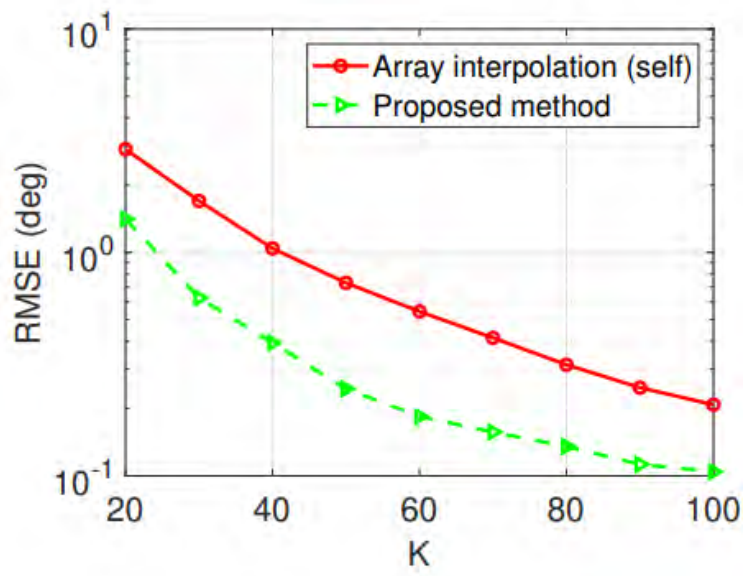
(a) Vs input SNR: 50 snapshots

(b) Vs number of snapshots: SNR=10 dB

In both case, using both auto- and cross-lags improve performance.

Note that, in the redundancy-free sparse array design, auto- and cross-lags differ.





[17] S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "DOA estimation exploiting interpolated multi-frequency sparse array," IEEE SAM Workshop, 2020.

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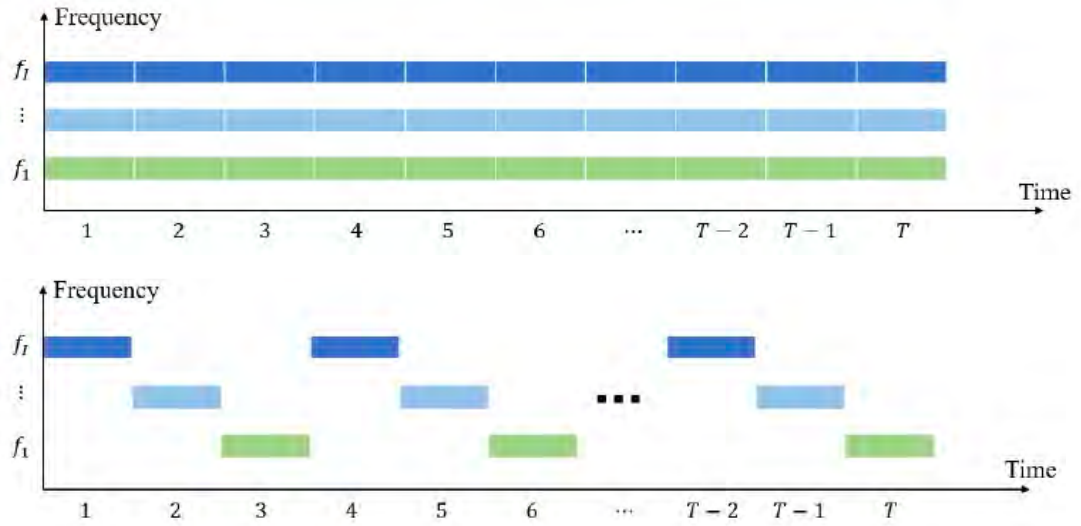
ASP Lab

Frequency-switching sparse array

- Multi-frequency sparse arrays require wideband processing of the received signal with a high complexity, making them infeasible for certain applications.
- Frequency-switching sparse arrays provide an effective alternative with similar performance [20].
- At any time, only a single-frequency component is processed.



- Multi-frequency: Power divided by I; all time samples used
- Frequency-selective: full power used; time samples divided by I

Processing

- Similar processing methods to the multi-frequency sparse arrays can be used to perform array interpolation and DOA estimation

[20] Y. D. Zhang and M. W. T. S. Chowdhury, "Frequency-switching sparse arrays," *IEEE SAM Workshop*, 2024.

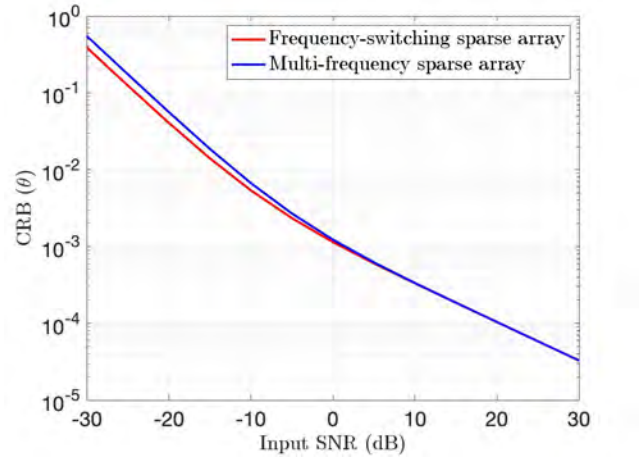
Frequency-switching sparse array

Cramer-Rao Bound (CRB) analysis

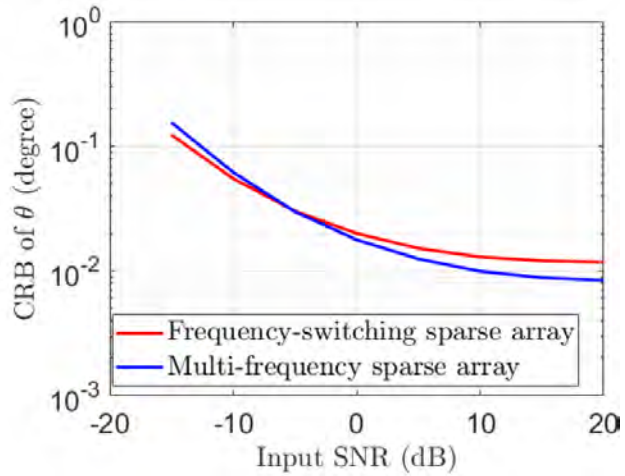
- Frequency-switches lowers the CRB at low SNR
- For over-determined DOA estimation, both sparse arrays achieve the same CRB at high SNR region
- For under-determined DOA estimation, multi-frequency sparse arrays achieve lower CRB at high SNR, due to the floor with the SNR

Example:

- 5 physical sensors; $I = 2$ frequencies
- $T = 5000$ snapshots
- 4 or 10 signals uniformly distributed in $[-60^o, 60^o]$



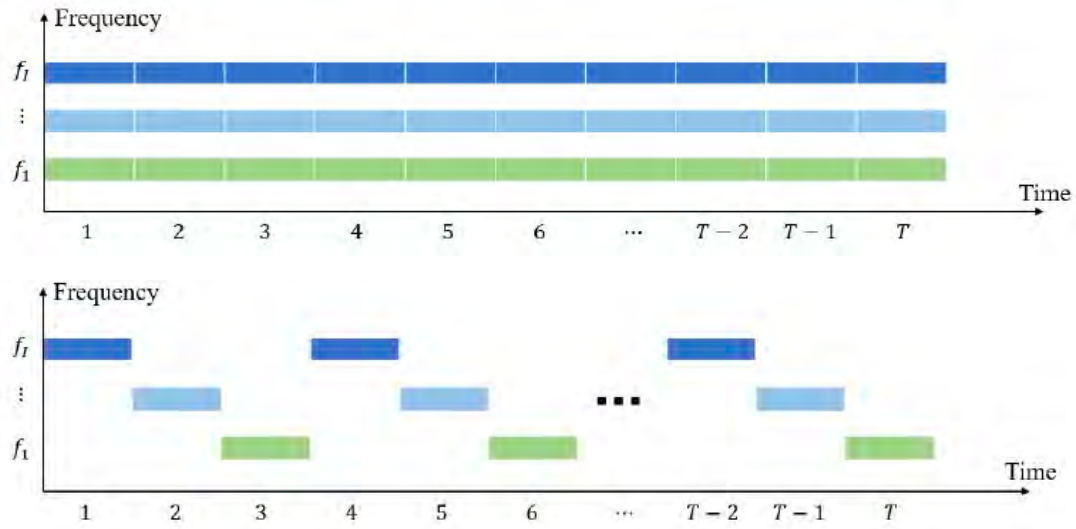
Overdetermined case (4 signals)

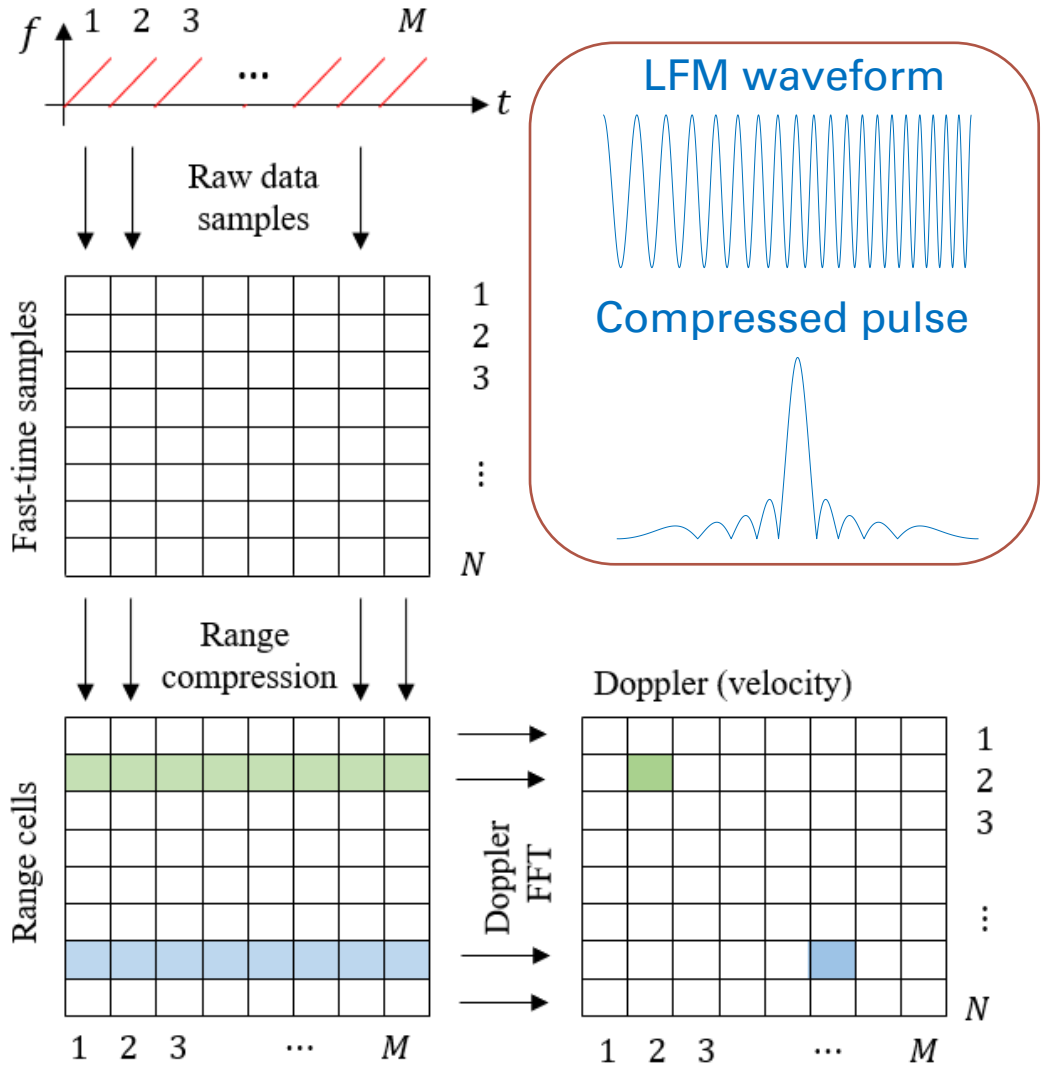


Underdetermined case (10 signals)

Frequency-switching sparse array

However, when we consider multi-frequency and frequency-switching in a slow-time framework, they yields the same power and DOA estimation performance.





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Remarks

- **Multi-frequency sparse array framework**
  - Provides significantly higher degrees of freedom and larger array apertures
  - Achieve unique lags in the order of I^2N^2 with N sensors and I frequencies
- **Key contributions:**
  - **Multiple frequencies:** Offer multiple virtual sub-arrays with larger array aperture and more DOFs
  - **Nonuniform physical array:** Reduce lag redundancies
  - **Modified array interpolation:** Enable robust DOA estimation
- **Open issues**
  - Understanding the offering of multi-frequency sparse arrays with close frequencies
    - Some preliminary results are reported for sparse arrays with compressed spacing
  - More effective processing of multi-frequency sparse array signals
  - Understand the dual-purpose use of spectrum for other sensing objectives
  - Extension to MIMO radar
  - Feasible hardware implementation

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