SPARSITY-BASED ROBUST ADAPTIVE BEAMFORMING EXPLOITING COPRIME ARRAY

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***Abstract*—In this paper, a novel sparsity-based adaptive beamforming algorithm is proposed to achieve effective interference cancellation using coprime arrays. To reconstruct the interference-plus-noise covariance matrix and obtain the steering vector of the desired signal required for robust beamforming, the power and directions-of-arrival (DOAs) of signals are estimated in the context of compress sensing. The results are then refined to obtain a more accurate estimation of the signal power so as to ensure effective interference cancellation. The power and DOA estimation is performed using the virtual array aperture of a coprime array in order to achieve improved estimation accuracy as compared to the results based directly on the physical array. The estimated power and DOA information are then used to reconstruct the interference-plus-noise covariance matrix and implement a robust adaptive beamformer. Simulation results demonstrate the effectiveness of the proposed algorithm.**

I. INTRODUCTION

Adaptive beamforming is a fundamental technique in array signal processing with wide applications in, e.g., radar, biomedical imaging, remote sensing, and radio astronomy [1]–[4]. Adaptive beamforming is the core technique that enables broad biomedical applications ranging from ultrasonic imaging to electroencephalogram (EEG), magnetoencephalogram (MEG), functional magnetic resonance imaging (fMRI), and magnetoencephalography [5]–[8].

Among the various adaptive beamformers developed in the literature, the minimum variance distortionless response (MVDR) beamfomer, also known as the Capon beamformer [9], is commonly used as it provides a high angular resolution and superior interference rejection capability. The MVDR beamformer, however, suffers from severe performance degradation when there is a model mismatch, e.g., when the assumed direction-of-arrival (DOA) of the desired signal is inaccurate or when the training interference data contains desired signal component. Therefore, designing robust adaptive beamformers against such mismatches is an important problem.

The interference-plus-noise covariance matrix reconstruction-based methods developed by Gu *et al* [10], [11] provide an efficient way to deal with such issue. In [10], the interference-plus-noise covariance matrix is reconstructed by integrating the Capon spectrum over an interference region, which is assumed to be separated from the desired signal direction. In [11], to reduce the computational complexity, the reconstruction operation is simplified to a summation form. Namely, the interference covariance matrix is expressed as a product sum of the outer product of the interference steering vector and its power. It is noted that the above methods are developed based on the uniform linear array (ULA) configuration.

Recently, sparse array configurations have attracted great attentions due to their many desirable features, such as larger aperture and increased number of degrees of freedom, as compared to the ULA counterpart [12]–[14]. In particular, among many sparse array configurations that are available in the literature, coprime array is attractive because of its capability for systematic design and analysis [15]–[18]. Coprime arrays take advantage of the difference coarray property to effectively convert the received signals associated with a physical array to those corresponding to a virtual array with a much larger aperture. As a result, coprime arrays have shown to achieve improved accuracy for DOA estimation and target location [19]–[25]. However, robust beamforming using coprime arrays in the presence of signal model mismatch has not been adequately studied.

In this paper, we develop a novel adaptive beamforming approach for coprime arrays that achieves accurate and robust adaptive beamforming. In this approach, the power and the DOAs of the interference signals are first estimated using the virtual array data. From the compressive sensing perspective [26], a two-step method is used to achieve accurate estimation of both the DOA and the associated power of all signals. Based on this estimate, the interference-plus-noise covariance matrix is reconstructed, free from the desired signal component, so as to achieve effective interference suppression and, at the same time, avoid the self-nulling problem of the desired signal. As the result, the proposed adaptive beamforming algorithm can achieve near-optimal output signal-plus-interference-noise ratio (SINR) performance. Especially, when the directions of sources are close to each other, the superiority of the proposed algorithm over existing adaptive beamformers becomes more pronounced.

Throughout this paper, we use lower-case and upper-case boldface characters to denote vectors and matrices, respectively. The superscripts $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ respectively denote the conjugate, transpose and conjugate transpose of a vector or matrix. $\|\cdot\|_2$ denotes the Euclidean norm of a vector, whereas $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively denote the l_0 and l_1 norms. In addition, $\text{vec}(\cdot)$ denotes the vectorization operator, $\otimes$ stands for the Kronecker product, and $\mathbf{I}$ denotes an identity matrix.

II. System Model of Coprime Array

We consider a coprime array consisting of a pair of uniform linear subarrays [16]. Denote the unit inter-sensor spacing as d which is half wavelength $\lambda/2$, and let M and N be a pair of coprime integers. Without loss of generality, it is assumed that the first subarray has $2M$ sensors with inter-sensor spacing Nd and the second subarray has N sensors with inter-sensor spacing Md. The first sensor of both subarrays coincides and is considered as the reference element. Therefore, there are $2M+N-1$ sensors in the coprime array.

Assume that Q uncorrelated, narrow-band and far-field signals impinging on the coprime array from distinct directions $\theta_1, \theta_2, \ldots, \theta_Q$. The received signal vector of the coprime array, $\mathbf{x}(k) \in \mathbb{C}^{(2M+N-1)\times 1}$, is expressed as

$$\mathbf{x}(k) = \sum_{q=1}^{Q} \mathbf{a}(\theta_q) s_q(k) + \mathbf{n}(k) = \mathbf{A}\mathbf{s}(k) + \mathbf{n}(k), \quad (1)$$

where $\mathbf{s}(k) = [s_1(k),\ s_2(k),\ \ldots,\ s_Q(k)]^T$, $\mathbf{a}(\theta_q) = [1, e^{j\frac{2\pi}{\lambda}p_2 \sin(\theta_q)}, \ldots, e^{j\frac{2\pi}{\lambda}p_{2M+N-1}\sin(\theta_q)}]^T$ is the steering vector of signal s_q with p_j denoting the array position of the jth sensor, $j = 1, 2, \ldots, 2M+N-1$, and $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_Q)]$ is the matrix consisting of the steering vectors corresponding to all signals. The elements of the noise vector $\mathbf{n}(k)$ follow the independent and identically distributed complex Gaussian distribution with mean 0 and variance σ_n^2. Without loss of generality, we assume that $s_1(k)$ and θ_1 are the desired signal and its DOA, respectively.

By applying adaptive beamforming to the coprime array with $\mathbf{w}(k) \in \mathbb{C}^{(2M+N-1)\times 1}$ as the weight vector, the output of the beamformer becomes

$$y(k) = \mathbf{w}^H(k)\mathbf{x}(k). \quad (2)$$

In the MVDR beamformer, the optimal weight vector is obtained by minimizing the array output variance while keeping the array response of the desired signal as 1, i.e. [9],

$$\min_{\mathbf{w}} \quad \mathbf{w}^H \mathbf{R}_{\text{i+n}} \mathbf{w} \qquad \text{s.t.} \qquad \mathbf{w}^H \mathbf{a}(\theta_1) = 1, \quad (3)$$

and its solution is given by

$$\mathbf{w}_{\text{opt}} = \frac{\mathbf{R}_{\text{i+n}}^{-1}\mathbf{a}(\theta_1)}{\mathbf{a}^H(\theta_1)\mathbf{R}_{\text{i+n}}^{-1}\mathbf{a}(\theta_1)}, \quad (4)$$

where $\mathbf{R}_{\text{i+n}}$ is the interference-plus-noise covariance matrix and can be approximately expressed as

$$\mathbf{R}_{\text{i+n}} = \sum_{j=2}^{Q} \sigma_j^2 \mathbf{a}(\theta_j)\mathbf{a}^H(\theta_j) + \sigma_n^2 \mathbf{I}_{2M+N-1}, \quad (5)$$

with σ_j^2 denoting the power of the jth interference signal. In practice, $\mathbf{R}_{\text{i+n}}$ is difficult to directly obtain and is often replaced by the sample covariance matrix

$$\hat{\mathbf{R}}_{\mathbf{xx}} = \frac{1}{L}\sum_{k=1}^{L} \mathbf{x}(k)\mathbf{x}^H(k), \quad (6)$$

where L is the number of snapshots. It is well known that $\hat{\mathbf{R}}_{\mathbf{xx}}$ is the maximum likelihood estimator of the theoretical covariance matrix $\mathbf{R}_{\mathbf{xx}}$ when L is sufficiently large. However, because $\hat{\mathbf{R}}_{\mathbf{xx}}$ contains the desired signal component, the beamforming performance would degrade, especially when the input signal-to-noise ratio (SNR) is high [10]. In order to avoid this problem and provide robust beamforming capability, we will consider accurate reconstruction of the interference-plus-noise covariance matrix, $\mathbf{R}_{\text{i+n}}$, in the following section.

III. Proposed Robust Beamformer

Classical methods for the reconstruction of the interference-plus-noise covariance matrix $\mathbf{R}_{\text{i+n}}$ include the integrating reconstruction [10] and the sparse reconstruction [11]. The integrating reconstruction approach makes use of the known angular region of the desired signals to estimate the interference-plus-noise covariance matrix through integrating the external of the desired signal region. The sparse reconstruction approach takes advantage of the sparsity of the sources in the spatial domain to reconstruct the covariance matrix of signals $\mathbf{R}_{\mathbf{xx}}$, and then remove the desired signal component from it. Moreover, the latter designed a two-step implementation structure, i.e., DOA estimation and power modification. Both approaches are developed based on a ULA and the latter approach has a lower computational complexity.

Different to these methods which deal with ULAs, the proposed algorithm considers an coprime array configuration and takes the sparsity of both the sources and the arrays into consideration. As such, it yields a novel robust beamforming algorithm that is developed through the estimation of the interference-plus-noise covariance matrix exploiting sparse reconstruction. Toward this end, an accurate estimation of the DOA and power of each signal is obtained from a virtual array, which are then used to reconstruct the steering vectors of the signals in the physical array [12]-[14]. In the following, we separately estimate the DOA and the power of the signals, and the proposed robust beamformer is presented.

Remark: compared with literature [11], there are two main differences. The first difference lies in the array structure. The method in [11] is based on ULA while the proposed algorithm is based on a coprime array. The second difference lies in the estimation method for DOAs and power of sources. The former utilizes the classical Capon spatial spectrum while the proposed algorithm makes use of the compressive sensing technique.

A. DOA Estimation

The DOA information of the signals is obtained in the virtual array which has a denser sensors as compared to the physical coprime array, and has a larger aperture when compared to the ULA with the same number of sensors. The virtual array output is obtained by vectorizing the sample covariance matrix, $\hat{\mathbf{R}}_{\mathbf{xx}}$, i.e.,

$$\mathbf{z} = \mathrm{vec}(\hat{\mathbf{R}}_{\mathbf{xx}}) = \tilde{\mathbf{A}}\mathbf{b} + \sigma_n^2\tilde{\mathbf{i}} + \boldsymbol{\epsilon}, \tag{7}$$

where $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}(\theta_1), \tilde{\mathbf{a}}(\theta_2), \ldots, \tilde{\mathbf{a}}(\theta_Q)]$ with $\tilde{\mathbf{a}}(\theta_q) = \mathbf{a}^*(\theta_q) \otimes \mathbf{a}(\theta_q)$ for $q = 1, 2, \cdots, Q$, $\mathbf{b} = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_Q^2]^T$, $\tilde{\mathbf{i}} = \mathrm{vec}(\mathbf{I}_{2M+N-1})$, and $\boldsymbol{\epsilon}$ represents discrepancies between $\mathbf{z}$ and the virtual array model. Vector $\mathbf{z}$ amounts to a single-snapshot signal vector received from the virtual array, whose steering matrix $\tilde{\mathbf{A}}$ represents an extended virtual aperture. In order to compute the optimal weight vector in (4), we need to estimate the DOA of the desired signal and the interference-plus-noise covariance matrix. The latter can be obtained from the power and directions of the interference signals. Toward this end, from the perspective of sparse signal recovery through compressing sensing, we perform the following optimization:

$$\begin{aligned} &\min_{\mathbf{b}^{\mathrm{o}}, \sigma_n^2} \| \mathbf{b}^{\mathrm{o}} \|_0 \\ \text{s.t.} \quad &\| \mathbf{z} - \tilde{\mathbf{A}}^{\mathrm{o}}\mathbf{b}^{\mathrm{o}} - \sigma_n^2\tilde{\mathbf{i}} \|_2 < \delta, \\ &\mathbf{b}^{\mathrm{o}}(j) \geq 0, j = 1,\ 2,\ \ldots,\ G, \end{aligned} \tag{8}$$

where δ is an adjustable parameter, $\tilde{\mathbf{A}}^{\mathrm{o}} = [\tilde{\mathbf{a}}(\theta_1), \tilde{\mathbf{a}}(\theta_2), \ldots, \tilde{\mathbf{a}}(\theta_G)]$ is the sensing matrix, and $G \gg Q$. In addition, $\mathbf{b}^{\mathrm{o}}$ is a sparse vector in which the positions of non-zero entries stand for the DOAs, whereas these non-zero values represent the corresponding signal power.

As equation (8) contains l_0 norm that makes the problem intractable, it is a common practice to approximate the l_0 norm by the l_1 norm. In so doing, equation (8) becomes the following basis pursuit denoising problem:

$$\begin{aligned} &\min_{\mathbf{b}^{\mathrm{o}}, \sigma_n^2} \| \mathbf{z} - \tilde{\mathbf{A}}^{\mathrm{o}}\mathbf{b}^{\mathrm{o}} - \sigma_n^2\tilde{\mathbf{i}} \|_2 + \tau \| \mathbf{b}^{\mathrm{o}} \|_1 \\ &\text{s.t.} \quad \mathbf{b}^{\mathrm{o}}(j) \geq 0, j = 1, 2, \ldots, G, \end{aligned} \tag{9}$$

where τ is a regularization parameter that trades off between the sparsity and the least square error. The above optimization is convex and can be solved by using linear programming techniques [27] [28].

Many compressive sensing methods, such as the least absolute shrinkage and selection operator (LASSO), can be used to solve the above optimization problem. These methods generally provide a good estimation of the DOA information, which is given as the positions of the non-zero elements in vector $\mathbf{b}^0$. Despite the accuracy in the estimated DOAs, however, the optimization problem formulated in (9) usually leads to an underestimated solution of the signal power. Such issue is often ignored in DOA estimation applications because the accuracy of estimated power is considered less important. In the underlying robust beamforming based on the reconstruction of the interference-plus-noise covariance matrix, however, such discrepancy becomes critical because any deviation in the estimated interference power will lead to an inaccurate interference-plus-noise covariance matrix and, in turn, a high residual interference power in the beamformer output. To ensure an accurate interference power estimation, we proposed a novel approach, as described below, to re-estimate the corresponding signal power after the DOA information is obtained using compressive sensing techniques like LASSO.

B. Power Estimation

For simplicity, we assume that noise power is known or is separately estimated [29]. Then, after obtaining the DOA information of the signals, we formulate the following least squares minimization problem:

$$\begin{aligned} &\min_{\mathbf{b}} \| \mathbf{z} - \bar{\mathbf{A}}\mathbf{b} - \sigma_n^2\tilde{\mathbf{I}} \|_2 \\ \text{s.t.} \quad &\mathbf{b}(j) > 0, j = 1, 2, \ldots, Q. \end{aligned} \tag{10}$$

In practice, as only the interference signals with a moderate strength are detected in the DOA estimation and need to be considered in robust beamforming, the entries of $\mathbf{b}$ are guaranteed to be positive. As such, we can ignore the inequality constraint, in (10), thus yielding the following closed-form least-squares solution:

$$\bar{\mathbf{b}} = (\bar{\mathbf{A}}^H\bar{\mathbf{A}})^{-1}\bar{\mathbf{A}}^H(\mathbf{z} - \sigma_n^2\bar{\mathbf{I}}), \tag{11}$$

where $\bar{\mathbf{A}} = [\tilde{\mathbf{a}}(\theta_1), \tilde{\mathbf{a}}(\theta_2), \ldots, \tilde{\mathbf{a}}(\theta_Q)]$, $\bar{\mathbf{b}} = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_Q^2]^T$. The estimated power spectrum of (8) is expressed as

$$\bar{\mathbf{b}}(\theta) = \begin{cases} \sigma_j^2, & \theta \in \theta_j, j = 1, 2, \ldots, Q, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

C. Robust Beamformer Design

Using the estimated Q-sparse power spectrum $\bar{\mathbf{b}}(\theta)$, the interference-plus-noise covariance of the physical coprime array is reconstructed as

$$\hat{\mathbf{R}}_{\text{i+n}} = \sum_{j=2}^{Q} \bar{\mathbf{b}}(\theta_j)\mathbf{a}(\theta_j)\mathbf{a}^H(\theta_j) + \sigma_n^2 \mathbf{I}_{2M+N-1}. \quad (13)$$

After estimating DOAs of all sources in (9), it is easy to determine $\mathbf{a}(\theta_1)$ using a priori angular region of the desired signal. Then, according to (4), the weight vector of the proposed adaptive beamformer is written as

$$\mathbf{w}_{\text{pro}} = \frac{\hat{\mathbf{R}}_{\text{i+n}}^{-1}\mathbf{a}(\theta_1)}{\mathbf{a}(\theta_1)^H \hat{\mathbf{R}}_{\text{i+n}}^{-1}\mathbf{a}(\theta_1)}. \quad (14)$$

It is worth noting that, in order to output the waveform of the desired signal, the beamformer and its associated weight vector must be designed based on the physical coprime array rather than the virtual array because, as we can clearly observe from (7), the latter only provides the power information of the signals whereas their waveform information is lost. That is, we estimate the DOAs and power of the signals from the virtual array for an higher accuracy, while the weight vector of the beamformer is obtained for the physical array where the waveform information of the signals is preserved.

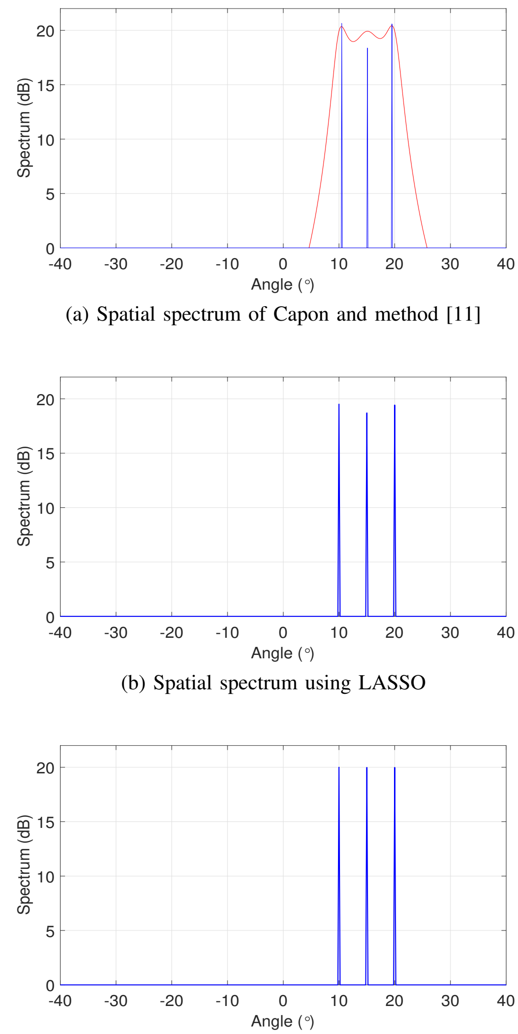
The proposed sparse reconstruction-based adaptive beamforming algorithm is summarized as follows:

Step 1: Estimate the sample covariance matrix using (6) and its vectorized result using (7).
Step 2: Obtain the signal DOAs using (9).
Step 3: Estimate the Q-sparse power spectrum using (11) and (12).
Step 4: Reconstruct the interference-plus-noise covariance matrix using (13).
Step 5: Compute the weight vector of the adaptive beamformer using (14).

IV. Simulation Results

Consider a coprime array equipped with 10 omnidirectional antenna sensors. The coprime integers are chosen as $M = 3$ and $N = 5$. The corresponding array sensors are respectively placed at $[0, 3, 6, 9, 12]d$ and $[0, 5, 10, 15, 20, 25]d$. We consider three uncorrelated, narrowband and far-field signals impinging on the coprime array. The noise vector is modeled as independent and identically distributed Gaussian random process. For each example, 500 Monte Carlo trials are carried out to obtain each simulation point.

The proposed algorithm is compared to the subspace-based method [4], the worst-case beamformer [2], the covariance matrix reconstruction method [10] and the sparse reconstruction method [11]. The sample grid is uniform distribution from $-40°$ to $40°$ with a $0.2°$ increment between adjacent grid points.



(a) Spatial spectrum of Capon and method [11]

(b) Spatial spectrum using LASSO

(c) Spatial spectrum using the proposed method

Fig. 1: Comparison of spatial spectra for three signals from $10°$, $15°$ and $20°$, each with a power of 20 dB above noise.

A. Spatial Power Spectrum Estimation

We first show the spatial spectrum obtained from the proposed algorithm, which is compared to that obtained from the sparse reconstruction method developed in [11], and the results where both the DOA and power are obtained using the LASSO. The DOA of the desired signal is $\theta_1 = 10°$ while those of the two interference signals are $\theta_2 = 15°$ and $\theta_3 = 20°$, respectively. The input signal-to-noise ratio (SNR) and input interference-to-noise ratio (INR) are both 20 dB, and the number of snapshots is fixed to 200. Note that the other considered beamformers are implemented without estimating the DOAs and power of the signals.

For the sparse reconstruction method developed in [11], the Capon spectrum (depicted as the red line in Fig. 1(a)) is first obtained with its peaks identifying the estimated signal DOAs, and the signal power is then estimated using a modified Capon estimator in these directions (depicted as blue lines in Fig. 1(a)). From Fig. 1(a), we observe that the Capon spectrum does not yield an accurate estimation of the signal DOAs due to the close angular separation and, as such, the subsequent signal power estimation is also inaccurate. In Fig. 1(b), the LASSO method can estimate the signal DOAs more accurately, but it tends to underestimate the signal power. On the other hand, as shown in Fig. 1(c), the proposed algorithm can estimate both DOAs and power with a high fidelity, thus leading to accurate reconstruction of the interference-plus-noise covariance matrix.

B. Adaptive Beamformer Design

We examine the beamformer performance by considering a scenario where the assumed direction of the desired signal is inaccurate. More specifically, the assumed DOA of the desired signal is $\hat{\theta}_1 = 14°$ while the actual one is $\theta_1 = 10°$. That is, there is a $4°$ mismatch in the direction of the desired signal. Other simulation parameters remain unchanged.

Fig. 2 compares the out SINR between the proposed method and the aforementioned methods. It is observed that the proposed algorithm achieves the best output SINR and the superiority becomes more pronounced as the input SNR increases. The covariance matrix reconstruction method and the sparse reconstruction approach cannot obtain an accurate interference-plus-noise covariance matrix because of the close angular separation of the interference signals, particularly when the input SNR is low. The worst-case beamformer and subspace-based approaches, on the other hand, fail to avoid the self-nulling problem of the desired signal because the sample covariance matrix includes the desired signal component. The proposed algorithm, on the other hand, offers near-optimal output SINR performance because it takes full advantages of the virtual aperture of the coprime array to achieve a high-fidelity estimation of both DOAs and power of all signals. These results are then used to reconstruct an accurate interference-plus-noise covariance matrix for effective interference cancelation while preserving the desired signal.

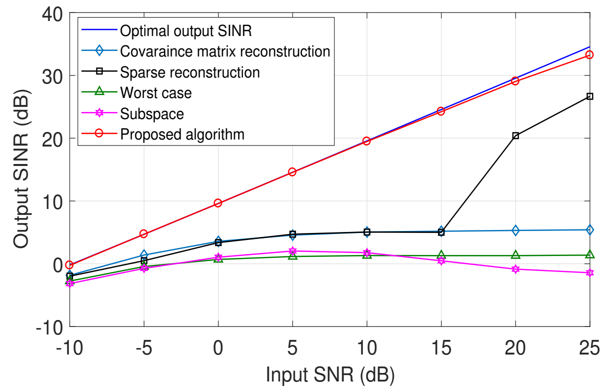


Fig. 2: Output SINR performance under look direction mismatch of the desired signal.

V. Conclusion

A novel robust adaptive beamforming algorithm against covariance matrix uncertainty was proposed in this paper. In order to construct an accurate interference-plus-noise covariance matrix, it is important to estimate the power and DOAs of the interference signals with a high accuracy. We developed a joint estimation method for the power and DOAs of all signals in a coprime array framework, and the problem is solved using compressive sensing techniques. As a result, improved beamforming performance was achieved even when the DOA of desired signal is close to those of interference signals. Simulation results clearly demonstrated that, compared with existing methods, the proposed algorithm provides a superior output SINR performance.

Acknowledgment

The work of K. Liu was supported by the China Scholarship Council for his stay at the Temple University. The work of Y. D. Zhang was supported in part by the National Science Foundation under Grant No. AST-1547420.

References

[1] J. Li and P. Stoica, *Robust Adaptive Beamforming*. John Wiley, 2005.

[2] S. A. Vorobyov, A. B. Gershman, and Z.-Q. Luo. “Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem,” *IEEE Trans. Signal Process.,* vol. 51, no. 2, pp. 313–324, 2003.

[3] H. Qian, K. Liu, and W. Wang, “Shrinkage widely linear recursive least square algorithms for beamforming,” *IEICE Trans. Commun.,* vol. 99, no. 7, pp. 1532–1540, 2016.

[4] D. D. Feldman and L. J. Griffiths, “A projection approach for robust adaptive beamforming,” *IEEE Trans. Signal Process.*, vol. 42, no. 4, pp. 867–876, 1994.

[5] A. Fuchs, “Beamforming and its applications to brain connectivity,” in V. K. Jirsa and A. R. McIntosh (ed.), *Handbook of Brain Connectivity*, Springer, 2007.

[6] M. Tanter and M. Fink, “Ultrafast imaging in biomedical ultrasound,” *IEEE Trans. Ultrason., Ferroelect., and Freq. Control*, vol. 61, no. 1, pp. 102–119, 2014.

[7] J. L. Semmlow and B. Griffel, *Biosignal and Medical Image Processing*. CRC press, 2014.

[8] F. Deligianni, M. Centeno, D. W. Carmichael, and J. D. Clayden, "Relating resting-state fMRI and EEG whole-brain connectomes across frequency bands," *Frontiers in Neuroscience*, vol. 8, no. 258, pp. 1–16, 2014.

[9] J. Capon, "High-resolution frequency-wavenumber spectrum analysis," *Proc. IEEE*, vol. 57, no. 8, pp. 1408–1418, 1969.

[10] Y. Gu and A. Leshem, "Robust adaptive beamforming based on interference covariance matrix reconstruction and steering vector estimation," *IEEE Trans. Signal Process.,* vol. 60, no. 7, pp. 3881–3885, 2012.

[11] Y. Gu, N. A. Goodman, S. Hong, and Y. Li, "Robust adaptive beamforming based on interference covariance matrix sparse reconstruction," *Signal Process.*, vol. 96, pp. 375–381, 2014.

[12] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, 2010.

[13] Y. D. Zhang, M. G. Amin, and B. Himed, "Sparsity-based DOA estimation using co-prime arrays," in *Proc. IEEE Int. Conf. Acousti., Speech and Signal Process. (ICASSP)*, Vancouver, Canada, May 2013, pp. 3967–3971.

[14] C. Zhou, Z. Shi, and Y. Gu, "Coprime array adaptive beamforming with enhanced degrees-of-freedom capability," in *Proc. IEEE Radar Conf.*, Seattle, WA, May 2017, pp. 1357–1361.

[15] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573-586, 2011.

[16] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the MUSIC algorithm," in *Proc. IEEE Digital Signal Processing Workshop and IEEE Signal Processing Education Workshop (DSP/SPE)*, Sedona, AZ, Jan. 2011, pp. 289–294.

[17] S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Process.,* vol. 63, no. 6, pp. 1377–1390, 2015.

[18] M. G. Amin, P. P. Vaidyanathan, Y. D. Zhang, and P. Pal, "Editorial for coprime special issue," *Digital Signal Process.*, vol. 61, pp. 1-2, 2017.

[19] C. Zhou, Z. Shi, Y. Gu, and X. Shen, "DECOM: DOA estimation with combined MUSIC for coprime array," in *Proc. Int. Conf. Wireless Commun. Signal Process.*, Hangzhou, China, Oct. 2013, pp.1–5.

[20] Q. Shen, W. Liu, W. Cui, S. Wu, Y. D. Zhang, and M. G. Amin, "Low-complexity wideband direction-of-arrival estimation based on co-prime arrays," *IEEE/ACM Trans. Audio, Speech and Language Process.*, vol. 23, no. 9, pp. 1445–1456, Sept. 2015.

[21] M. Guo, T. Chen, and B. Wang, "An improved DOA estimation approach using coarray interpolation and matrix denoising," *Sensors*, vol. 17, no. 1140, 2017.

[22] S. Qin, Y. D. Zhang, and M. G. Amin, "DOA estimation of mixed coherent and uncorrelated targets exploiting coprime MIMO radar," *Digital Signal Process.*, vol. 61, pp. 26–34, 2017.

[23] S. Qin, Y. D. Zhang, M. G. Amin, and F. Gini. "Frequency diverse coprime arrays with coprime frequency offsets for multitarget localization," *IEEE J. Selected Topics in Signal Process.*, vol. 11, no. 2, pp. 321–335, 2017.

[24] S. Qin, Y. D. Zhang, M. G. Amin, and B. Himed, "DOA estimation exploiting a uniform linear array with multiple co-prime frequencies," *Signal Process.*, vol. 130, pp. 37–46, 2017.

[25] C. Zhou, Y. Gu, Y. D. Zhang, Z. Shi, T. Jin, and X. Wu, "Compressive sensing-based coprime array direction-of-arrival estimation," *IET Commun.*, vol. 11, no. 11, pp. 1719–1724, Sept. 2017.

[26] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inform. Theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[27] D. P. Bertsekas, *Nonlinear Programming*. Athena Scientific, 1999.

[28] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge Univ. Press, 2004.

[29] K. Harmanci, J. Tabrikian, and J. L. Krolik, "Relationships between adaptive minimum variance beamforming and optimal source localization," *IEEE Trans. Signal Process.*, vol. 48, no. 1, pp. 1–12, 2000.