Effective Nested Array Design for Fourth-Order Cumulant-Based DOA Estimation

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***Abstract*—In this paper, we propose a novel sparse array design for direction-of-arrival estimation based on the fourth-order statistics of the received signals. The utilization of fourth-order statistics provides a higher number of degrees of freedom and flexibility as compared to the commonly used covariance-based methods. The proposed sparse array design scheme yields the fourth-order difference co-array which offers a significant increment in the number of consecutive lags as compared to the existing sparse array structures used by fourth-order cumulant-based direction-of-arrival estimation methods. Moreover, the proposed array is designed in such a way that the resulting difference co-array does not contain any holes. Simulation results verify the effectiveness of the proposed array structure.**

keywords: Sparse array, direction-of-arrival estimation, fourth-order difference co-array.

I. Introduction

Direction-of-arrival (DOA) estimation plays an important role in radar, sonar, wireless communications, radio surveillance, and several other applications [1, 2]. While subspace-based methods that use second-order (SO) statistics are commonly used for DOA estimation, the exploitation of higher-order statistics (HOS) for DOA estimation offers a higher number of degrees of freedom (DOF) so that the DOAs of more sources than the number of array sensors can be determined. In this context, the cumulant-based virtual array concept and its variants have received broad interests [3–5]. Because Gaussian random variables have zero HOS, HOS-based DOA estimation methods also provide the capability of eliminating Gaussian random components, such as thermal noise, so as to enhance signals that exhibit non-Gaussian characteristics [6].

Detection of more sources than the number of sensors can also be achieved by using a sparse array in the context of co-arrays [7]. The minimum redundancy array (MRA) [8] is a well known linear sparse array structure that achieves the maximum number of consecutive lags in the yielding difference co-array. The minimum hole array (MHA) [9] is another sparse array configuration which minimizes the number of holes in the difference co-array which is a highly desired property. However, as the array configurations of MRA and MHA structures cannot be analytically expressed, systematic design and precise DOF analysis become difficult. Recently, great efforts have been paid to develop sparse array structures that can be systematically designed and analyzed. Two notable sparse array configurations of this type include the nested array (NA) [10] and the co-prime array (CPA) [11]. The NA configuration uses two uniform linear sub-arrays such that one of the sub-arrays has unit inter-element spacing. On the other hand, CPA utilizes two linear sub-arrays such that the number of elements in each sub-array is a coprime pair and the inter-element spacing of each sub-array is proportional to the number of elements in the other sub-array. The achievable number of DOFs in terms of the consecutive and unique lags of these arrays and the generalized array structures have been well studied [12]. These efforts exploit the improved array aperture and DOFs offered by the difference co-arrays so as to resolve more sources and achieve improved estimation accuracy. Various DOA estimation methods were applied to such sparse arrays based on subspace and compressive sensing techniques [11–17].

The application of fourth-order cumulant (FOC) sparse sensor arrays for DOA estimation has recently attracted great attention as it provides further DOF increment to resolve even more sources. For example, the extension of four-level (4-L) and three-level (3-L) NA concept to the case of FOC was investigated in [18] and [19], respectively. Moreover, 3-L FOC-based difference co-array of CPA was discussed in [20]. The focus of these approaches was mainly the extension of array aperture and DOFs by increasing the number of sub-arrays. It is noted that, in these FOC-based sparse array processing, little attention has been paid towards the optimization of the inter–sub-array and inter-element spacings. While the inter–sub-array and inter-element spacings have been analyzed and optimized in generalized co-prime arrays under the SO statistics [12], vital changes in the array structure are required in the FOC-based array processing because there is a significant difference between SO and fourth-order (FO) statistics based formation of difference co-arrays. As such, the array design should be thoroughly re-investigated to reduce the number of holes and to increase the number of unique as well as consecutive lags in the resultant difference co-arrays.

In this paper, we propose a novel two-level (2-L) NA structure for FOC-based DOA estimation which results in a hole-free FO difference co-array, i.e., the resulting difference co-array will consist of virtual sensors such that there is no gap to break the consecutiveness of the co-array. The proposed sparse array structure effectively utilizes the inter-element and inter–sub-array spacings for the underlying 2-L NA for the

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case of FO statistics. For a practical scenario utilizing a small number of sensors, the proposed 2-L nested array structure provides a significantly higher number of consecutive lags compared to the existing 3-L and 4-L sparse array structures when FO statistics are exploited. Moreover, the promising no-hole feature in the resulting difference co-array makes the proposed structure an ideal one for a wide variety of DOA estimation algorithms. Simulation results illustrate the effectiveness of the proposed array structure.

II. Problem Formulation

A. Signal Model

Consider Q narrowband uncorrelated non-Gaussian stationary signals impinging on a P-sensor sparse linear array from angles $\theta_1, \theta_2, \ldots, \theta_Q$. The $P \times 1$ received signal vector $\mathbf{x}(t)$ at the sensor array is expressed as:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad (1)$$

where $\mathbf{A}$ is the $P \times Q$ array manifold matrix, $\mathbf{s}(t)$ represents the $Q \times 1$ signal vector impinging on the antenna array, $\mathbf{n}(t)$ is the $P \times 1$ noise vector that follows the complex white Gaussian distribution with zero mean and covariance matrix $\sigma_n^2\mathbf{I}$, and $\mathbf{I}$ is the $P \times P$ identity matrix.

B. Fourth-Order Statistics

Several definitions of FOCs exist in literature. In this paper, we utilize the following definition for the calculation of FOC:

$$\begin{aligned} c(l) =& \text{cum}\{x_{q_1}, x_{q_2}^*, x_{q_3}, x_{q_4}^*\} \\ =& \mathrm{E}\left\{x_{q_1}x_{q_2}^*x_{q_3}x_{q_4}^*\right\} - \mathrm{E}\left\{x_{q_1}x_{q_2}^*\right\}\mathrm{E}\left\{x_{q_3}x_{q_4}^*\right\} \\ &- \mathrm{E}\left\{x_{q_1}x_{q_4}^*\right\}\mathrm{E}\left\{x_{q_3}x_{q_2}^*\right\} \\ &- \mathrm{E}\left\{x_{q_2}^*x_{q_4}^*\right\}\mathrm{E}\left\{x_{q_1}x_{q_3}\right\}, \end{aligned} \quad (2)$$

where x_{q_1}, x_{q_2}, x_{q_3} and x_{q_4} denote the received signals at the q_1-th, q_2-th, q_3-th, and q_4-th array sensors, respectively, $\text{cum}(\cdot)$ is the cumulant operator, $(\cdot)^*$ is the conjugate operator, and $l = q_2 + q_4 - q_1 - q_3$. Note that the mixed use of conjugated and unconjugated elements in FOC computation yields both sum co-array and difference co-array sensors in the resultant virtual array.

Denoting K as the number of positive lags obtained from the FOC computation. The $(K+1) \times (K+1)$ cumulant matrix $\mathbf{C}$ can be computed using the vector of $2K+1$ consecutive lags estimated using Eq. (2) as follows:

$$\mathbf{C} = \begin{bmatrix} c(0) & c(1) & \cdots & c(K) \\ c(-1) & c(0) & \cdots & c(K-1) \\ \vdots & \vdots & \ddots & \\ c(-K) & c(-K+1) & \cdots & c(0) \end{bmatrix}. \quad (3)$$

Since the cumulants for Gaussian noise are zero, from Eqs. (2) and (3), we have:

$$\mathbf{C} = \tilde{\mathbf{A}}\mathbf{S}\tilde{\mathbf{A}}^H. \quad (4)$$

Here, $\tilde{\mathbf{A}}$ is $(K+1) \times Q$ extended array manifold matrix and $\mathbf{S} = \text{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_Q\}$, where γ_k is the kurtosis of the kth source signal.

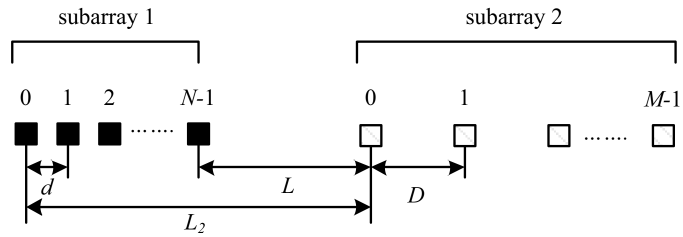


Fig. 1. The proposed sparse array configuration

III. Proposed Array Structure

A. Notations

Consider a sparse linear array consisting of two uniform linear sub-arrays as shown in Fig. 1, and let set $\mathcal{S}$ contain the individual locations of each array sensor that are physically present. Assume that the first sub-array consists of N sensors with an inter-element spacing of $d = \lambda/2$, where λ denotes the wavelength, whereas the second sub-array consists of M sensors with an inter-element spacing of D such that $M \geq N$. The linear displacement between the two sub-arrays (inter–sub-array spacing) is set as L. Moreover, we assume that both D and L are integer multiples of d.

The locations of sensors in sub-array 1 and sub-array 2 can be represented by the sets of locations $\mathcal{J}$ and $\mathcal{K}$, respectively, such that:

$$\mathcal{J} = \{l | l = kd, 0 \leq k \leq N-1\}, \quad (5)$$

and

$$\mathcal{K} = \{l | l = kD + L_2, 0 \leq k \leq M-1\}, \quad (6)$$

where $L_2 = (N-1)d + L$.

Our objective is to design a sparse array such that the resulting FO difference co-array has a high number of consecutive lags and, at the same time, contains no holes. For this purpose, we derive the mathematical relationships between parameters D, L and the maximum achievable number of lags.

B. Determination of Parameters D and L

Let us consider the FO co-arrays resulting from the array geometry under consideration. We start with the SO difference co-array of $\mathcal{J}$. Denote $\ominus$ and $\oplus$, respectively, as the SO difference and sum co-array operators, respectively [7]. From Eq. (5), the SO difference and sum co-arrays of $\mathcal{J}$, respectively, can be written as follows:

$$\mathcal{J} \ominus \mathcal{J} = \{l | l = kd, -(N-1) \leq k \leq N-1\}, \quad (7a)$$

$$\mathcal{J} \oplus \mathcal{J} = \{l | l = kd, 0 \leq k \leq 2(N-1)\}. \quad (7b)$$

From Eqs. (5) and (7), the third-order and FO difference co-arrays of $\mathcal{J}$ have the virtual sensors positioned at:

$$\mathcal{J} \ominus \mathcal{J} \ominus \mathcal{J} = \{l | l = kd, -2(N-1) \leq k \leq N-1\}, \quad (8)$$

$$\mathcal{J} \ominus \mathcal{J} \oplus \mathcal{J} \ominus \mathcal{J} = \{l | l = kd, -2(N-1) \leq k \leq 2(N-1)\}. \quad (9)$$

From Eq. (9), the FO difference co-array of $\mathcal{J}$ results in $4N-3$ consecutive elements with an inter-element spacing of d. It can be observed in Eqs. (6) and (8) that, if the inter-element spacing is set as $D = (3N-2)d$, then $\mathcal{K} \ominus \mathcal{J} \oplus \mathcal{J} \ominus \mathcal{J}$ will result in a uniform linear array with sensors in $\mathcal{J} \oplus \mathcal{J} \ominus \mathcal{J}$

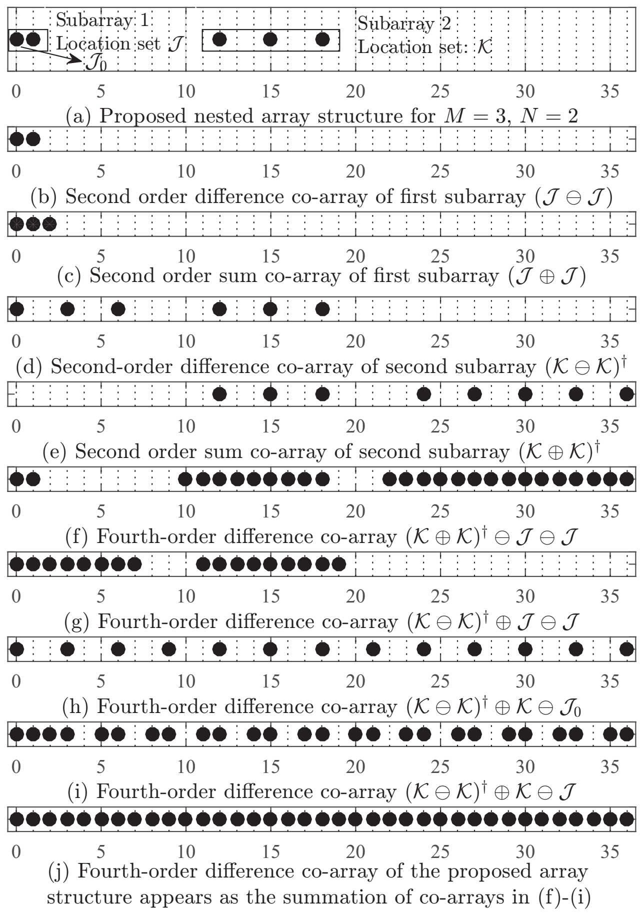


Fig. 2. A step-wise illustration of non-negative co-array lags for the proposed array design $(N = 2, M = 3)$.

wrapped around each element of sub-array 2. However, this choice of D will result in holes at other locations in the FO difference co-array.

In the following, we consider the optimal selection of parameters D and L such that the resulting FO difference co-array does not contain any holes. For notational simplicity, we will consider in the sequel only the non-negative portion of the difference co-arrays due to their conjugate-symmetry property. The virtual sensors in the SO difference co-array of the second sub-array $\mathcal{K}$ are located at:

$$\mathcal{K} \ominus \mathcal{K} = \{l | l = kD, 0 \leq k \leq M-1\}, \tag{10}$$

which expresses the difference co-array due to the difference in the location of elements in the sub-array 2. If the actual sensor positions are also included as the SO difference co-array between $\mathcal{K}$ and the first sensor (located at origin $\mathcal{J}_0$) of sub-array $\mathcal{J}$, we can write:

$$(\mathcal{K} \ominus \mathcal{K})^{\dagger} = \{l | l = kD, 0 \leq k \leq M-1\} \cup \mathcal{K}. \tag{11}$$

On the other hand, the elements in the sum co-array of sub-array 2 has the following locations:

$$\mathcal{K} \oplus \mathcal{K} = \{l | l = kD + 2L_2, 0 \leq k \leq 2(M-1)\}, \tag{12}$$

which is the set of co-array locations due to the sum in the location of elements in sub-array 2. Similarly, if the actual elements of $\mathcal{K}$ are included as the SO sum co-array between $\mathcal{K}$ and the 0th sensor of sub-array $\mathcal{J}$, we have:

$$(\mathcal{K} \oplus \mathcal{K})^{\dagger} = \{l | l = kD + 2L_2, 0 \leq k \leq 2(M-1)\} \cup \mathcal{K}. \tag{13}$$

If $D = (2N-1)d$, from Eqs. (7) and (11), it can be observed that:

$$(\mathcal{K} \ominus \mathcal{K})^{\dagger} \oplus \mathcal{J} \ominus \mathcal{J} = \{l | l = k_1 d\} \cup \{l | l = L_2 + k_2 d\}, \tag{14}$$

where $0 \leq k_1 \leq (M-1)(2N-1) + N - 1$ and $-N + 1 \leq k_2 \leq (M-1)(2N-1) + N - 1$. Similarly,

$$(\mathcal{K} \oplus \mathcal{K})^{\dagger} \ominus \mathcal{J} \ominus \mathcal{J} = \{l | l = k_1 d + L_2\} \cup \{l | l = 2L_2 + k_2 d\}, \tag{15}$$

where $2N - 1 \leq k_1 \leq (M-1)(2N-1)$ and $-2(N-1) \leq k_2 \leq 2(M-1)(2N-1)$. Analyzing Eqs. (14) and (15), it is clear that the two co-arrays $(\mathcal{K} \ominus \mathcal{K})^{\dagger} \oplus \mathcal{J} \ominus \mathcal{J}$ and $(\mathcal{K} \oplus \mathcal{K})^{\dagger} \ominus \mathcal{J} \ominus \mathcal{J}$ have the holes in the positive half only at two locations which are from $(2N-1)(M-1) + N - 1$ to $L_2 - (2N-1)$ and from $L_2 + 2(N-1)M$ to $2L_2 - 2(N-1)$. $(\mathcal{K} \oplus \mathcal{K})^{\dagger} \ominus \mathcal{K} \ominus \mathcal{J}$ can be used to fill these gaps; however, this co-array consists of several sub-arrays of length N such that we can only use one N-element patch from this co-array at each of the above-mentioned gaps and we can fill these gaps perfectly if and only if the following relation is held:

$$L = (2MN + N - M)d. \tag{16}$$

This completes our objective of no-hole two-level sparse array with the value of D determined as

$$D = (2N-1)d. \tag{17}$$

C. Determination of Number of Lags

Let us re-write Eq. (6) in terms of the newly determined values of parameters D and L, i.e.,

$$\mathcal{K} = \{l | l = (2MNd + 2(k+1)N - M - k - 1)d\}, \tag{18}$$

where $0 \leq k \leq M - 1$ and the last element of $\mathcal{K}$ should be at location $(4MN - 2M)d$. Let $\mathcal{S} = \mathcal{J} \cup \mathcal{K}$ denote the entire sensor array location set. The virtual elements in the FOC difference co-array of $\mathcal{S}$ will have the locations at $\mathcal{S} \ominus \mathcal{S} \oplus \mathcal{S} \ominus \mathcal{S}$. Now, from Eq. (18), the last element of the FO difference co-array of the entire array structure $(\mathcal{S} \ominus \mathcal{S} \oplus \mathcal{S} \ominus \mathcal{S})$ should be at location $(8MN - 4M)d$. Since the resulting FO difference co-array is symmetric and does not contain any hole, the total number of elements in the FO difference array $\mathcal{S} \ominus \mathcal{S} \oplus \mathcal{S} \ominus \mathcal{S}$ is $16MN - 8M + 1$. Fig. 2 shows one such array configuration for $M = 3$ and $N = 2$ and the corresponding sum and difference co-arrays.

Table I illustrates the comparison of different sparse array structures and the total number of consecutive lags for the corresponding FO difference co-arrays, whereas Table II compares the number of consecutive lags for each co-array structure where the total number of physical sensors is chosen to be $5, 9, 11$, and 12. It can be observed that the proposed array structure yields the greatest number of consecutive lags for $M + N < 12$. It is important to note that [18, 19] and [20] mainly focus on increasing the number of sub-arrays while ignoring the DOF provided by the effective utilization of

TABLE I. COMPARISON OF DIFFERENT SPARSE ARRAY STRUCTURES FOR FOC-BASED DOA ESTIMATION

Array structure	No. of sensors	No. of consecutive lags
4-L FO-NA [18]	$\sum_{i=1}^{4} M_i + 1$	$2\prod_{i=1}^{4}(M_i+1) + 2\prod_{i=1}^{3}(M_i+1) - 1$
3-L FO-NA [19]	$\sum_{i=1}^{3} M_i$	$2(3M_3+2)M_2(M_1+1) - 4M_3 - 3$
3-L FO-CPA [20]	$\sum_{i=1}^{3} M_i - 1$	$4M_1M_2M_3 + 3M_1M_2 + 2M_1M_3 - 2M_2M_3 + M_1 - 2M_2 + 2M_3 - 1$
Proposed 2-L FO-NA	$M+N$	$16MN - 8M + 1$

inter-element and inter–sub-array spacings. However, we have used the optimized inter-element and inter–sub-array spacings which result in an increased DOF for a small number of sensors for the case of a 2-L NA structure. Moreover, for the sparse array with 11 sensors, the proposed array structure provides 441 consecutive lags, greater than the consecutive lags provided by any of the other structures, which is sufficient for many practical applications.

Since we have $16MN - 8M + 1$ consecutive lags in the FO difference co-array for the sparse array design considered in Fig. 1, Eq. (3) provides the cumulant matrix of order $(K+1) \times (K+1)$, where $K = 8MN - 4M + 1$. After the calculation of the cumulant matrix, subspace based methods like MUSIC can be readily exploited for efficient DOA estimation. The MUSIC spectrum can be calculated using:

$$P_{MUSIC}(\theta_k) = \frac{1}{\mathbf{a}^H(\theta_k)\mathbf{V}\mathbf{V}^H\mathbf{a}(\theta)}, \tag{19}$$

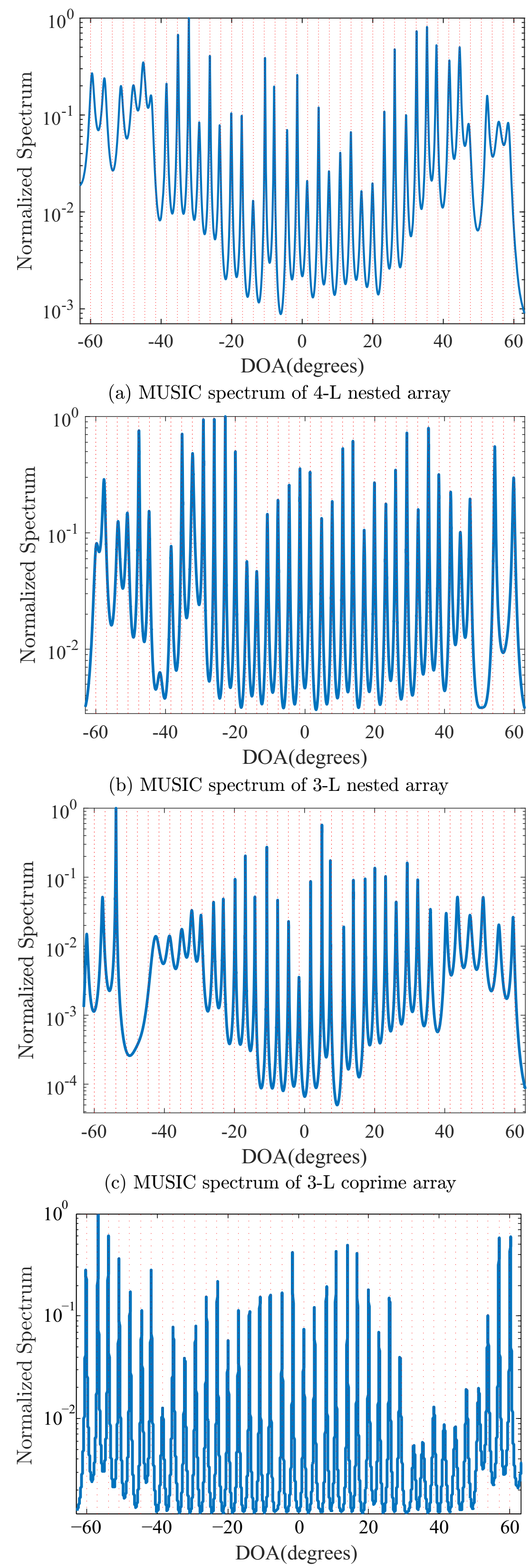
where $\mathbf{V}$ is the noise subspace consisting of $K-Q$ eigenvectors of $\mathbf{C}$ corresponding to its $K-Q$ least significant eigenvalues and $\mathbf{a}(\theta)$ is the steering vector in the direction of angle θ_k.

IV. SIMULATION RESULTS

In this section, we present simulation results to demonstrate the performance of the proposed array structure and compare it with existing methods. We consider 40 independent sources, uniformly distributed between -60^o and 60^o, impinging on the sparse linear array consisting of 7 sensors $(M=3, N=4)$.

TABLE II. NUMBER OF CO-ARRAY LAGS ACHIEVED BY DIFFERENT SPARSE ARRAY STRUCTURES FOR FOC-BASED DOA ESTIMATION

Array Structure	No. of sensors	Configuration	No. of consecutive lags
4-L FO-NA [18]	5	$(1,1,1,1)$	47
3-L FO-NA [19]	5	$(1,2,2)$	53
Proposed 2-L FO-NA	5	$(2,3)$	81
4-L FO-NA [18]	9	$(2,2,2,2)$	215
3-L FO-NA [19]	9	$(3,3,3)$	249
3-L FO-CPA [20]	9	$(4,3,3)$	189
Proposed 2-L FO-NA	9	$(4,5)$	289
4-L FO-NA [18]	11	$(3,3,2,2)$	383
3-L FO-NA [19]	11	$(4,3,4)$	401
2-L FO-CPA [20]	11	$(4,3,5)$	293
Proposed 2-L FO-NA	11	$(5,6)$	441
4-L FO-NA [18]	12	$(3,3,3,2)$	511
3-L FO-NA [19]	12	$(4,4,4)$	541
3-L FO-CPA [20]	12	$(6,4,3)$	375
Proposed 2-L FO-NA	12	$(6,6)$	529



(a) MUSIC spectrum of 4-L nested array

(b) MUSIC spectrum of 3-L nested array

(c) MUSIC spectrum of 3-L coprime array

(d) MUSIC spectrum of the proposed 2-L nested array

Fig. 3. Normalized MUSIC spectrum for the proposed and existing DOA estimation techniques [18–20] (DOA = 40 sources uniformly distributed from -60^o to 60^o, 500,000 snapshots, noise-free).

Noise-free conditions are considered and $500,000$ snapshots are used for DOA estimation. The FO difference co-arrays used in [18–20] and the proposed structure were considered having $107, 133, 85$ and 169 consecutive lags, respectively. For DOA estimation, only consecutive lags are considered and the MUSIC algorithm has been applied to the resulting spatial smoothing matrix of estimated cumulants.

It can be observed from Fig. 3 that, by using the existing array designs, we are not able to resolve all the 40 sources. On the other hand, all the sources are successfully resolved if the proposed sparse array structure is used. This illustrates the improved resolution capability and higher number of DOFs offered by the proposed 2-L nested array structure.

V. Conclusion

In this paper, we have presented a novel two-level nested sparse array configuration for the case of fourth-order cumulants based DOA estimation which utilizes the optimized inter-element and inter–sub-array spacings. The array is designed to yield a non-hole co-array with the maximum number of virtual sensors. Simulation results show that the proposed array configuration outperforms the existing array configurations by resolving more sources in the case of less number of sensors. Moreover, the absence of any holes in the difference co-array makes the proposed configuration attractive for a variety of DOA estimation algorithms.

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