Coprime Array-based DOA Estimation in Unknown Nonuniform Noise Environment

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Abstract

In this paper, we propose direction-of-arrival (DOA) estimation techniques, respectively based on covariance matrix reconstruction and matrix completion, to achieve robust DOA estimation capability in nonuniform noise environments using coprime arrays. For the covariance matrix reconstruction-based approach, by exploring the diagonal structure of the covariance matrix of the noise, the covariance matrix of the received signal vector is reconstructed through averaging its diagonal elements. Moreover, in order to handle more sources than the number of sensors, the difference coarray of coprime arrays is utilized through the vectorization of the reconstructed covariance matrix. A compressive sensing (CS) based DOA estimator is then formulated to provide sparsity-based DOA estimation. For the matrix completion-based approach, we take the full advantage of the difference coarray lags and obtain the noise-free covariance matrix of the virtual uniform linear array by using the matrix completion technique to recover the removed diagonal elements and missing holes in the virtual array covariance matrix. Then CS-based and MUSIC-based DOA estimators are respectively designed to perform DOA estimation using the estimated noise-free covariance matrix. Simulation results verify the effectiveness of the proposed algorithms.

Index Terms

Direction-of-arrival estimation, nonuniform noise, coprime array, sparse array, compressive sensing.

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I. Introduction

Direction-of-arrival (DOA) estimation is a vital task in array signal processing and is widely used in various applications, such as radar, biomedical imaging, remote sensing, and radio astronomy [1–3]. In the past decades, various DOA estimation methods, such as MUSIC [4], ESPRIT [5] and compressive sensing (CS) [6] [7], were developed to obtain a high-accuracy performance. Most of the existing DOA estimation techniques are developed under the assumption that the noise is uniform, that is, the noise power is the same at each sensors. However, in some practical situations, the noise may become nonuniform, yielding an arbitrary diagonal noise covariance matrix. In this case, the performance of DOA estimator will degrade. For example, for subspace-based DOA estimation methods, such as MUSIC and ESPRIT, eigen-decomposition of the data covariance matrix does not lead to correct signal and noise subspace estimation. Similarly, for CS-based DOA estimation methods, it becomes difficult to accurately recover the direction information.

So far, a number of methods have been developed in the literature to provide robust DOA estimation in the presence of nonuniform noise [8–19]. In [9], for example, the covariance matrix of nonuniform noise is first estimated from the array covariance matrix, and high-performance DOA estimation is achieved after pre-whitening the data. However, it requires the number of sensors to be at least three times higher than the number of source. Two maximum likelihood (ML)-based DOA estimators are derived in [10] [11], which use an iterative manner to estimate the signal DOAs and noise parameter. These methods suffer a high computational load. Two subspace estimation-based methods are proposed in [16] to estimate the signal and noise subspaces. However, they require an iterative procedure and thus are time-consuming. After setting diagonal elements of the covariance matrix to be a same value, the signal and noise subspaces can be directly estimated through eigen-decomposition [17]. By using the matrix completion technique, an iteration-free method developed in [18] can offer satisfactory performance. However, the number of resolvable sources is less than the number of sensors.

On the other hand, to achieve a higher number of degrees-of-freedom (DOFs) than the number of physical sensors, a number of sparse linear arrays have been developed. For a given number of physical sensors, the minimal redundancy arrays (MRA) yields a maximum number of consecutive virtual arrays [20]. However, the MRA does not have a closed-form expression about its physical array locations and the achievable number of DOFs. Recently, the nested array

[21] and the coprime array [22] are developed as alternative sparse array designs that permit systematical design with known sensor positions and analytical expression for the achievable number of unique and consecutive lags [23]. Both nested and coprime arrays consist of two uniform linear subarrays. For the nested array, one of the subarrays has a half-wavelength spacing, and all the virtual sensors in the resulting difference coarray are always consecutive. A nested array detects $O(N^2)$ sources with $O(N)$ physical sensors. Compared with the nested array configure, the interelement spacing of the two subarrays used in a coprime array have a coprime relationship. A coprime array is more flexible in achieving a large array aperture and reducing the mutual coupling effects, thus attracted considerable interests (e.g., [23–31]). Unlike the nested array, the virtual sensors in the resulting difference coarray constructed from a coprime array are generally non-consecutive, i.e., there are holes between the virtual sensors.

Similar to the case in uniform linear arrays (ULAs), nonuniform noise could lead to severe performance degradation of the DOA estimators in a coprime array. Our study interests focus on the design of novel DOA estimators that work robustly in nonuniform noise environment. It is noted that there are holes in the difference coarray from a coprime array. In this case, CS-based methods do not need to consider the continuity issue of the virtual array sensors by building the dictionary matrix through vectorization of the array covariance matrix. For subspace-based methods, e.g., MUSIC and ESPRIT, spatial smoothing has to be used to restore the rank of the virtual sensor covariance matrix and thus only the consecutive lags can be used [32, 33]. The method developed in [12] is a simple and efficient CS-type DOA estimation algorithm in the presence of nonuniform noise. The results have been extended to wideband signals in [13]. However, these methods directly remove the diagonal elements from the data covariance matrix, thus resulting in a performance loss. Recently, reference [19] proposed a DOA estimation method for coprime arrays and it could be regarded as a sparse array version of method described in [17]. However, the method in [19] therein only uses the consecutive lags, and any non-consecutive lags are discarded. Therefore, it generally does not achieve the best possible performance as coprime arrays usually generate coarrays with holes.

In this paper, we develop new methods, respectively based on covariance matrix reconstruction and matrix completion, to achieve effective DOA estimation for coprime arrays in the presence of nonuniform noise. For the method based on covariance matrix reconstruction, in order to decrease the effect of nonuniform noise, the array covariance matrix is reconstructed to follow a Hermitian structure with equal diagonal elements. Then, by exploring the position structure of the vectored array covariance matrix, the DOA estimation problem is converted to a sparse signal recovering problem which is solved through convex optimization methods. For matrix completion-based approach, in order to remove nonuniform noise covariance and take a full advantage of difference coarray lags, we first remove the diagonal element of array covariance, and matrix completion technique [34]–[35] is then used to recover the values corresponding to the holes in difference coarray and the diagonal elements of the virtual covariance matrix. In this approach, because the restored virtual array is uniform linear, we use both CS and MUSIC methods to estimate the source DOAs.

The remainder of this paper is organized as follows. In Section II, we describe the signal model of coprime arrays in the presence of unknown nonuniform noise. In Section III, the proposed covariance matrix reconstruction-based and matrix completion-based methods are respectively presented. The Cramér-Rao bound (CRB) existence conditions of the proposed algorithm are derived in the Section IV. In Section V, simulation results are provided to demonstrate the effectiveness and superiority of the proposed DOA estimation methods in terms of detection performance, angular resolution, and estimation accuracy. Section VI concludes this paper.

Throughout this paper, we use lower-case and upper-case boldface characters to denote vectors and matrices, respectively. The superscripts $(\cdot)^*$, $(\cdot)^{\mathrm{T}}$ and $(\cdot)^{\mathrm{H}}$ respectively stand for the conjugate, transpose and conjugate transpose of a vector or matrix. $||\cdot||_2$ denotes the Euclidean (l_2) norm of a vector, whereas $||\cdot||_0$ and $||\cdot||_1$ respectively denote the l_0 and l_1 norms. $||\cdot||_*$ denotes the nuclear norm. $\circ$, $\otimes$ and $\odot$ stand for the Hadamard product, the Kronecker product and the Khatri-Rao product, respectively. In addition, $\mathrm{vec}(\cdot)$ is the vectorization operator, and $\mathrm{diag}\{\mathbf{a}\}$ denotes a diagonal matrix whose diagonal elements are the elements of $\mathbf{a}$. In addition, $\mathrm{trace}(\mathbf{A})$ denotes the trace of the matrix $\mathbf{A}$. $\mathrm{E}[\cdot]$ is the expectation operator. $\mathbf{I}_N$ represents the $N \times N$ identity matrix. $[\mathbf{x}_{\mathbb{S}}]_j$ denotes the j entry of $[\mathbf{x}_{\mathbb{S}}]$, and $\langle \mathbf{x}_{\mathbb{S}} \rangle_i$ denotes the value on the support $i \in \mathbb{S}$. For example, if $\mathbb{S} = \{2, 4, 6\}$ and $[\mathbf{x}_{\mathbb{S}}] = \{1, 3, 5\}$, then $[\mathbf{x}_{\mathbb{S}}]_1 = 1$, $[\mathbf{x}_{\mathbb{S}}]_2 = 3$, $[\mathbf{x}_{\mathbb{S}}]_3 = 5$, and $\langle \mathbf{x}_{\mathbb{S}} \rangle_2 = 1$, $\langle \mathbf{x}_{\mathbb{S}} \rangle_4 = 3$ and $\langle \mathbf{x}_{\mathbb{S}} \rangle_6 = 5$.

II. Signal Model

Let M and N be a pair of coprime integers. Consider a coprime array which consists of two uniform linear subarrays. The two subarrays have $2M$ and N sensors, respectively, and their respective element spacing is Nd and Md. The unit interelement spacing is d which is set to half wavelength, denoted as $\lambda/2$. Without loss of generality, we assume $2M > N$. We assume

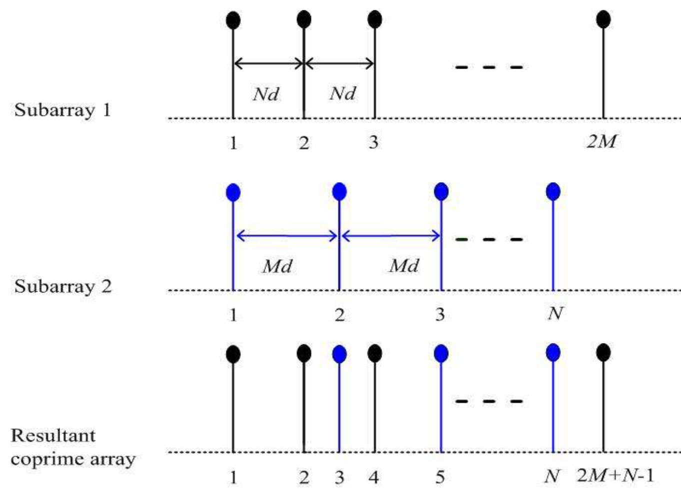

Fig. 1: Array geometry of coprime array

that the first sensor of both subarrays coincides and is set as the reference element. Due to the coprime relationship between M and N, except the reference sensor, other sensors do not overlap with each other. Hence, the sensor positions of the difference coarray $\mathbf{p} = [p_1, p_2, ..., p_{2M+N-1}]^{\mathrm{T}}$ are located at

$$\mathbb{S} = \{Mnd, 0 \leq n \leq N-1\} \cup \{Nmd, 0 \leq m \leq 2M-1\}, \tag{1}$$

and there are a total number of $|\mathbb{S}| = 2M + N - 1$ sensors in the coprime array. The array geometry of coprime array is shown in Fig.1.

Assume L far-field uncorrelated narrow-band signals impinging on the coprime array from L distinct angles $\theta_1, \theta_2, \ldots, \theta_L$. The $(2M+N-1)$-dimensional received signal vector $\mathbf{x}(k)$ is expressed as

$$\mathbf{x}(k) = \sum_{l=1}^{L} \mathbf{a}(\theta_l) s_l(k) + \mathbf{n}(k) = \mathbf{A}\mathbf{s}(k) + \mathbf{n}(k), \tag{2}$$

where k is the index of sample snapshot, $\mathbf{s}(k) = [s_1(k),\ s_2(k),\ \ldots,\ s_L(k)]^{\mathrm{T}}$ is the source signal vector, $\mathbf{a}(\theta_l) = [1, e^{\jmath \frac{2\pi}{\lambda} p_2 \sin(\theta_l)}, \ldots, e^{\jmath \frac{2\pi}{\lambda} p_{2M+N-1} \sin(\theta_l)}]^{\mathrm{T}}$ is the signal steering vector with p_g, $g = 1, 2, \ldots, 2M+N-1$, being the sensor position, and $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_L)]$ is the steering matrix. $\mathbf{n}(k)$ is the additive complex Gaussian noise vector with uncorrelated entries.

The covariance matrix of the received signal vector $\mathbf{x}(k)$ is expressed as

$$\mathbf{R_{xx}} = \mathrm{E}\{\mathbf{x}(k)\mathbf{x}^{\mathrm{H}}(k)\} = \mathbf{A}\mathbf{P}\mathbf{A}^{\mathrm{H}} + \mathbf{Q}, \tag{3}$$

where $\mathbf{P} = \mathrm{E}\{\mathbf{s}(k)\mathbf{s}^{\mathrm{H}}(k)\}$ and $\mathbf{Q} = \mathrm{E}\{\mathbf{n}(k)\mathbf{n}^{\mathrm{H}}(k)\}$ are the signal covariance matrix and the noise covariance matrix, respectively. It is common to assume $\mathbf{n}(k)$ as a Gaussian white noise vector with identical power, and the noise covariance matrix is expressed as $\mathbf{Q} = \sigma_n^2 \mathbf{I}_{2M+N-1}$, where σ_n^2 is the noise power at each sensor. In this paper, however, we consider a nonuniform Gaussian noise environment, where the noise power in each sensor is different and unknown. In this case, the noise covariance matrix becomes

$$\mathbf{Q} = \mathrm{diag}\{\Psi_1^2, \Psi_2^2, ..., \Psi_{2M+N-1}^2\}, \tag{4}$$

where $\Psi_g^2, g = 1, ..., 2M+N-1$, denotes the noise power measured at the gth sensor. Clearly, the uniform noise environment can be considered as a special case of the nonuniform noise. In practice, the array covariance matrix is estimated by

$$\hat{\mathbf{R}}_{\mathbf{xx}} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{x}(k)\mathbf{x}^{\mathrm{H}}(k), \tag{5}$$

where K denotes the number of sample snapshots.

III. Proposed DOA Estimation Algorithms

To mitigate the effect of unknown nonuniform noise to the DOA estimation performance and take full advantage of the virtual array aperture of the coprime array, in this paper, three DOA estimators are proposed for effective DOA estimation. All these techniques explore the intrinsic structure of the covariance matrix of the received signal vector and the unknown noise variance to effectively mitigate the effect of nonuniform noise covariance entries. They can be classified into two major categories, one based on covariance matrix reconstruction and two based on matrix completion. (a) In the covariance matrix reconstruction method, the sample covariance matrix is first reconstructed to maintain a Hermitian structure with equal diagonal elements. Then, the DOA estimation problem is converted to a sparse signal recovery problem which can be solved efficiently by convex optimization techniques. (b) In the matrix completion methods, the nonuniform denoising problem is first converted to signal subspace estimation problem. We then rebuild a noise-free covariance matrix using the matrix completion technique, and estimate the signal DOAs respectively using CS and MUSIC-based methods.

A. Proposed Method 1: Covariance matrix reconstruction based CS DOA estimator

It is well known that the nonuiformity of noise will result in performance degradation, especially in a low SNR condition. According to (3), the covariance matrix of nonuniform noise is a diagonal matrix with unequal diagonal entries. Therefore, to eliminate the nonuiformity in

$\mathbf{Q}$, a simple method is to reconstruct the sample covariance matrix $\hat{\mathbf{R}}_{\mathbf{xx}}$ by averaging its main diagonal elements. As a result, the new covariance matrix $\mathbf{R}_1$ is expressed as

$$\mathbf{R}_1 = \mathbf{J}_1 \circ \hat{\mathbf{R}}_{\mathbf{xx}} + \eta \mathbf{I}_{2M+N-1}, \tag{6}$$

where $\mathbf{J}_1 \in \mathbb{C}^{(2M+N-1)\times(2M+N-1)}$ is an all-one matrix except the diagonal elements which are all zeros, and η is the averaged value of the diagonal elements of the sample covariance matrix $\hat{\mathbf{R}}_{\mathbf{xx}}$, i.e.,

$$\eta = \frac{1}{2M+N-1}\text{trace}(\hat{\mathbf{R}}_{\mathbf{xx}}). \tag{7}$$

To take advantage of the virtual array of the coprime array to estimate more signal DOAs than the number of physical sensors, we vectorize $\mathbf{R}_1$, yielding

$$\mathbf{z} = \text{vec}(\hat{\mathbf{R}}_1) = \tilde{\mathbf{A}}\mathbf{b} + \eta\tilde{\mathbf{i}} + \boldsymbol{\epsilon}, \tag{8}$$

where $\tilde{\mathbf{A}} = [\text{vec}(\mathbf{J}_1)\circ\tilde{\mathbf{a}}(\theta_1), \text{vec}(\mathbf{J}_1)\circ\tilde{\mathbf{a}}(\theta_2), \ldots, \text{vec}(\mathbf{J}_1)\circ\tilde{\mathbf{a}}(\theta_L)]$ is a steering matrix with $\tilde{\mathbf{a}}(\theta_l) = \mathbf{a}^*(\theta_l)\otimes\mathbf{a}(\theta_l)$ for $l = 1, 2, \cdots, L$, $\mathbf{b} = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_L^2]^{\text{T}}$ is the signal power, and $\tilde{\mathbf{i}} = \text{vec}(\mathbf{I}_{2M+N-1})$ is the vectorized identity matrix. $\boldsymbol{\epsilon}$ represents discrepancies between $\mathbf{z}$ and the virtual array model. As such, $\mathbf{z}$ amounts to a single-snapshot received vector corresponding to a virtual array with an extended dimension. We can use the CS technique to utilize the coarray aperture for effective DOA estimation. In this approach, we build an over-complete sensing matrix through grid sampling over the potential angular region, e.g., from -90° to 90°. Then, by using a sparse signal recovering method through CS, we obtain a sparse vector representing the signed power in each grid point, and the non-zero entries correspond to the source DOAs.

Denote $\theta_1^o, \theta_2^o, ..., \theta_G^o$ as G samples grid points in the angular region, where $G \gg L$, and construct $\tilde{\mathbf{A}}_1 = [\text{vec}(\mathbf{J}_1)\circ\tilde{\mathbf{a}}(\theta_1^o), \text{vec}(\mathbf{J}_1)\circ\tilde{\mathbf{a}}(\theta_2^o), \ldots, \text{vec}(\mathbf{J}_1)\circ\tilde{\mathbf{a}}(\theta_G^o)]$ as the sensing matrix. Then, the cost function of the DOA estimation problem is expressed as

$$\min_{\mathbf{b}_1} \parallel \mathbf{b}_1 \parallel_0 \quad \text{s.t.} \quad \parallel \mathbf{z} - \tilde{\mathbf{A}}_1\mathbf{b}_1 - \eta\tilde{\mathbf{i}} \parallel_2 < \delta, \tag{9}$$

where δ is a user-specific parameter. Because the l_0-norm problem in (9) is non-convex, we relax the above l_0 norm to l_1 norm and use the Lasso approach for sparse signal recovery. The Lasso cost function is expressed as

$$\min_{\mathbf{b}_1} \parallel \mathbf{z} - \tilde{\mathbf{A}}_1\mathbf{b}_1 - \eta\tilde{\mathbf{i}} \parallel_2 + \tau \parallel \mathbf{b}_1 \parallel_1 \tag{10}$$

where τ is a regularization parameter that trades off between the sparsity and the accuracy. The optimization problem (10) is convex and can be solved by, e.g., the interior-point method [36].

In this paper, we assume that the true DOAs are on the sampling grid, and extension to grid-off cases is straightforward [37]. For the sake of convenience, the covariance matrix reconstruction based DOA estimator is named as Proposed Method 1.

B. Proposed Method 2: Matrix completion based CS DOA estimator

Proposed Method 1 can suppress nonuniform noise through averaging the diagonal elements of the sample covariance matrix. In essence, it is to whiten the covariance matrix and mitigate the effect of nonuniform noise. In the following two sections, we use the matrix completion technique to recover a noise-free covariance matrix, and CS and MUSIC-based methods are respectively used to perform DOA estimation.

For the sake of clarify, several definitions are borrowed from [35] as follows:

Definition 1 (difference coarray integer set $\mathbb{D}$) For a sparse array where physical sensor positions formulate an integer set $\mathbb{S}$, whose difference coarray is given as $\mathbb{D} = \{c_1 - c_2 | c_1, c_2 \in \mathbb{S}\}$.

Definition 2 (maximum contiguous lag integer set $\mathbb{U}$) The maximum contiguous segment in $\mathbb{D}$ is given as $\mathbb{U} = \{c | \{-|c|, ..., -1, 0, 1, ..., |c|\} \subseteq \mathbb{D}\}$.

Definition 3 (shortest contiguous ULA integer set $\mathbb{V}$) The shortest contiguous segment containing $\mathbb{D}$ is given as $\mathbb{V} = \{c | \min(\mathbb{D}) \leq c \leq \max(\mathbb{D})\}$.

Definition 4 (weight function of coprime array w(c)) $\mathbb{Z} \to \mathbb{Z}$: $w(c) = |\{(c_1, c_2) \in \mathbb{S}^2 | c_1 - c_2 = c\}|$ which means the appearance times of each element in difference coarray $\mathbb{D}$.

Definition 5 (transfer matrix $\mathbf{F} \in \mathbb{Z}^{|\mathbb{D}| \times |\mathbb{S}|^2}$) A transfer matrix is used to remove the repeated information in the difference coarray and is defined as $\langle \mathbf{F} \rangle_c = \text{vec}(\mathbf{H}(c))^T$, where $\mathbf{H}(c) \in \mathbb{R}^{|\mathbb{S}| \times |\mathbb{S}|}$ satisfies

$$\langle \mathbf{H}(c) \rangle_{c1,c2} = \begin{cases} \dfrac{1}{w(c)}, \text{ if } c_1 - c_2 = c, \; c_1, c_2 \in \mathbb{S}, \\ 0, \qquad \text{otherwise}. \end{cases} \tag{11}$$

Matrix completion is an efficient method to recover a low-rank matrix from a finite sample set of entries. From (3), we know that the unknown covariance matrix of the nonuniform noise only affects the diagonal entries of the covariance matrix of the received signals. Therefore, unlike Proposed Method 1 which averages the diagonal elements of the sample covariance matrix, the proposed matrix completion based methods delete the diagonal elements and recover the entire noise-free covariance matrix of the virtual ULA defined by the integer set $\mathbb{V}$, including those corresponding to missing holes. Removing the diagonal entries of the sample covariance matrix yields

$$\mathbf{R}_2 = \mathbf{J}_1 \circ \hat{\mathbf{R}}_{\mathbf{xx}}. \tag{12}$$

After vectorizing $\mathbf{R}_2$ and rearranging the results through a transfer matrix $\mathbf{F}$, the new received vector based on the virtual ULA is expressed as

$$\mathbf{z}_2 = \mathbf{F}\text{vec}(\hat{\mathbf{R}}_2), \tag{13}$$

where $\mathbf{z}_2$ amounts to a single-snapshot received signal. Note that a covariance matrix constructed from $\mathbf{z}_2$ is rank one because only a single snapshot is available. Spatial smoothing is often utilized to restore its rank. However, the spatial smoothing operator can only deal with contiguous lags. In this case, to fill the holes in the difference coarray and form a virtual ULA in $\mathbb{V}$, the matrix completion technique is used to recover the missing elements.

A low-complexity direct spatial smoothing method is designed in [33] which, in addition to achieving spatial smoothing similar to [32], also fill in holes through matrix completion. The following virtual covariance matrix, which includes missing entries, is obtained by arranging the entries of the virtual received signal vector,

$$\mathbf{R}_3 = \begin{pmatrix} \langle \mathbf{z}_2 \rangle_0 & \langle \mathbf{z}_2 \rangle_{-1} & \cdots & \langle \mathbf{z}_2 \rangle_{-M_\xi} \\ \langle \mathbf{z}_2 \rangle_1 & \langle \mathbf{z}_2 \rangle_0 & \cdots & \langle \mathbf{z}_2 \rangle_{-M_\xi+1} \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{z}_2 \rangle_{M_\xi} & \langle \mathbf{z}_2 \rangle_{M_\xi-1} & \cdots & \langle \mathbf{z}_2 \rangle_0 \end{pmatrix} \tag{14}$$

where $M_\xi = \max(\mathbb{D})$ is the largest array aperture of virtual ULA.

To interpolate the holes and diagonal elements in $\mathbf{R}_3$ and recover noise-free covariance matrix $\mathbf{R}_4$ in virtual ULA, the coarray interpolation method performs the following optimization:

$$\begin{aligned} &\min_{\mathbf{R}_4} \quad \text{rank}(\mathbf{R}_4) \\ \text{s.t.}\ &[\mathbf{R}_4]_{c1,c2} = [\mathbf{R}_3]_{c1,c2},\ c_1 \neq c_2, c_1 - c_2 \notin \mathbb{V} - \mathbb{D}, \\ &\mathbf{R}_4 = \mathbf{R}_4^{\mathrm{H}}. \end{aligned} \tag{15}$$

Compared to [33], we added the constraint $c_1 \neq c_2$ because the diagonal elements are removed and need to be re-estimated. However, the above matrix rank optimization is NP-hard. Convex relaxation is usually applied to yield the following nuclear norm minimization problem

$$\begin{aligned} &\min_{\mathbf{R}_4} \quad \| \mathbf{R}_4 \|_* \\ \text{s.t. } &[\mathbf{R}_4]_{c1,c2} = [\mathbf{R}_3]_{c1,c2}, \; c_1 \neq c_2, c_1 - c_2 \notin \mathbb{V} - \mathbb{D}, \\ &\mathbf{R}_4 = \mathbf{R}_4^{\mathrm{H}}. \end{aligned} \tag{16}$$

As $\mathbf{R}_4$ is a positive-semidefinite Hermitian matrix, the nuclear norm can be further written as

$$\| \mathbf{R}_4 \|_* = \text{trace}(\mathbf{R}_4), \tag{17}$$

and the constraint $[\mathbf{R}_4]_{c_1,c_2} = [\mathbf{R}_3]_{c_1,c_2}, c_1 \neq c_2, c_1 - c_2 \notin \mathbb{V} - \mathbb{D}$ can be simplified as

$$\mathbf{J}_2 \circ [\mathbf{R}_4 - \mathbf{R}_3] = 0, \tag{18}$$

where $\mathbf{J}_2$ is an $((M_\xi + 1) \times (M_\xi + 1))$-dimensional projection matrix which is an all-one matrix except when the coordinate of matrix $[\mathbf{J}_2]_{c_1,c_2}$ satisfies $c_1 = c_2$ or $c_1 - c_2 \in \mathbb{V} - \mathbb{D}$ where the values are set to zero. Then, considering the covariance matrix error, the optimization problem (16) can be written as

$$\begin{aligned} &\min_{\mathbf{R}_4} \quad \text{trace}(\mathbf{R}_4) \\ \text{s.t. } &\| \mathbf{J}_2 \circ [\mathbf{R}_4 - \mathbf{R}_3] \|_2 \leq \delta_1, \\ &\mathbf{R}_4 = \mathbf{R}_4^{\mathrm{H}}, \end{aligned} \tag{19}$$

where δ_1 is a positive constant representing the estimation accuracy. It takes a small positive value when the number of snapshots is sufficiently large. Both the cost function and the constraints are convex. Therefore, (19) can be solved efficiently by using a convex optimization toolbox [36]. After obtaining $\mathbf{R}_4$, CS and MUSIC can be used to perform DOA estimation. The resulting algorithms using CS and MUSIC are respectively referred to as Proposed Method 2 and Proposed Method 3, and are described below.

Similar to Proposed Method 1, the CS technique can be used to estimate the DOAs from $\mathbf{R}_4$. One choice would be to utilize directly the covariance matrix $\mathbf{R}_4$ to design the DOA estimator as proposed in reference [38]. However, for low-complexity implementation, we only use the first column of $\mathbf{R}_4$, denoted as $\mathbf{r}_4$, to construct the following sparse signal recovery problem

$$\min_{\mathbf{b}_2} \| \mathbf{b}_2 \|_0 \quad \text{s.t.} \quad \| \mathbf{r}_4 - \tilde{\mathbf{A}}_2 \mathbf{b}_2 \|_2 < \delta_2, \tag{20}$$

where δ_2 is a user-specific parameter, $\tilde{\mathbf{A}}_2 = [\mathbf{a}_{\mathbb{V}}(\theta_1^{\mathrm{o}}), \mathbf{a}_{\mathbb{V}}(\theta_2^{\mathrm{o}}), \ldots, \mathbf{a}_{\mathbb{V}}(\theta_G^{\mathrm{o}})]$ is the sensing matrix, and $\mathbf{a}_{\mathbb{V}}(\theta^{\mathrm{o}}) = [1, e^{\jmath \frac{2\pi}{\lambda} d \sin(\theta^{\mathrm{o}})}, \ldots, e^{\jmath \frac{2\pi}{\lambda} (M_\xi) d \sin(\theta^{\mathrm{o}})}]^{\mathrm{T}}$ is the steering vector of the virtual ULA. We use Lasso to solve the following optimization problem

$$\min_{\mathbf{b}^2} \parallel \mathbf{r}_4 - \tilde{\mathbf{A}}_2 \mathbf{b}_2 \parallel_2 + \tau_2 \parallel \mathbf{b}_2 \parallel_1, \tag{21}$$

where τ_2 is a regularization parameter similar to (10). It is noted that the sensing matrix in (21) is different to that in (10), because $\tilde{\mathbf{A}}_2$ corresponds to a fully augmented ULA. The matrix completion based CS algorithm is named as Proposed Method 2.

C. Proposed Method 3: Matrix completion based MUSIC DOA estimator

Compared with the CS-based method, the MUSIC algorithm is a classical alternative to perform effective DOA estimation. As the reconstructed virtual array is uniform linear, the MUSIC algorithm achieves a similar number of DOFs to the CS-based method. The eigen-decomposition of the estimated covariance matrix $\hat{\mathbf{R}}_4$ results in

$$\hat{\mathbf{R}}_4 = \mathbf{V}_s \mathbf{\Phi}_s \mathbf{V}_s^{\mathrm{H}} + \mathbf{V}_n \mathbf{\Phi}_n \mathbf{V}_n^{\mathrm{H}}, \tag{22}$$

where matrix $\mathbf{V}_s$ contains the eigenvectors corresponding to the L largest eigenvalues in $\mathbf{\Phi}_s$ and spans the signal subspace, whereas $\mathbf{V}_n$ contains the eigenvectors corresponding to the $M_\xi - L + 1$ smallest eigenvalues in $\mathbf{\Phi}_n$ and spans the noise subspace. The L highest peaks of the following pseudo spatial spectrum correspond to the signal DOAs:

$$p(\theta) = \frac{1}{\mathbf{a}_{\mathbb{V}}^{\mathrm{H}}(\theta) \mathbf{V}_n \mathbf{V}_n^{\mathrm{H}} \mathbf{a}_{\mathbb{V}}(\theta)}. \tag{23}$$

The matrix completion-based MUSIC algorithm is referred to as Proposed Method 3.

The last two proposed DOA estimation methods based on matrix completion (Method 2 and Method 3) are summarized as follows:

Step 1: Estimate the sample covariance matrix using (5) and remove the diagonal elements using (12).

Step 2: Rearrange sample covariance matrix and obtain received signal vector of the virtual ULA using (13).

Step 3: Obtain the covariance matrix of the virtual array by optimizing (19).

Step 4: For Proposed Method 2, we use (21) to obtain the signal DOAs through Lasso. For Proposed Method 3, we first construct the noise subspace of $\hat{\mathbf{R}}_4$ using (22) and then perform DOA estimation using (23).

Both the matrix reconstruction-based method (Method 1) and the matrix completion-based methods (Methods 2 and 3) solve the covariance matrix denoising problem to mitigate the effect of the nonuniform noise. For proposed Method 1, we enforce the diagonal elements of the covariance matrix to take the same value. For proposed Methods 2 and 3, the matrix completion technique is used to recover the noise-free low-rank virtual covariance matrix. As all proposed algorithms can effectively suppress the effect of nonuniform noise, they provide a similar performance. Compared with Method 1, Methods 2 and 3 have a lower computational complexity as the transfer matrix $\mathbf{F}$ in (11) reduces the dimension of the virtual received signal vector. In particular, Method 3 is subspace-based and thus does not need to consider the off-grid problem. On the other hand, in a low SNR case, Method 1 achieves slightly better performance than Methods 2 and 3, because the latter suffers from performance degradation in matrix completion when the input SNR is low.

IV. CRB And Its Existence Condition

In this section, we derive the CRB and its existence condition in the presence of nonuniform noise. The derivation process is similar to that in [12], [39] and [40], but we prove the nonsingular condition of Fish information matrix (FIM) in the nonuniform noise condition. Therefore, the CRB expression is different to those obtained in [39] and [40], as those results are derived under the assumption of uniform Gaussian white noise, and are not applicable to the underlying situation with nonuniform noise. Note that the CRB expression derived here is equivalent to that in [12] derived for nested arrays, but we derive the results in a different form so as to facilitate the analysis of the CRB existence condition. Reference [12] did not provide the CRB existence condition.

The CRB of the DOAs in nonuniform noise environment is expressed as

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{1}{K}\{\boldsymbol{\Gamma}^{\mathrm{H}}[\mathbf{I}-\mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}]\boldsymbol{\Gamma}\}^{-1}. \tag{24}$$

where the matrix $\boldsymbol{\Gamma}$ is given as

$$\boldsymbol{\Gamma} = \jmath 2\pi(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}}\otimes\mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\mathbf{A}_{w}\mathbf{P}, \tag{25}$$

$\mathbf{R}_{\mathbb{S}}$ is the covariance matrix of the physical array, $\mathbf{r}_{\mathbb{S}} = \mathrm{vec}(\mathbf{R}_{\mathbb{S}})$, $\mathbf{P} = \mathrm{diag}(\sigma_1^2, ..., \sigma_L^2)$, $\mathbf{A}_w = \{\mathbf{A}_{\mathbb{S}}^{*}\odot[\mathrm{diag}(\mathbb{S})\mathbf{A}_{\mathbb{S}}]+[\mathrm{diag}(\mathbb{S})\mathbf{A}_{\mathbb{S}}^{*}]\odot\mathbf{A}_{\mathbb{S}}\}$, and the matrix $\mathbf{D}$ is given as

$$\mathbf{D} = (\mathbf{R}_{\mathbb{S}}^{\mathrm{T}}\otimes\mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\mathbf{W}, \tag{26}$$

where $\mathbf{W} = [\mathbf{A}_{\mathbb{S}}^* \odot \mathbf{A}_{\mathbb{S}},\ \mathbf{I}_{|\mathbb{S}|} \odot \mathbf{I}_{|\mathbb{S}|}]$. The derivation process of CRB expression is given in Appendix A. In the following two theorems, we provide the nonsingular condition of the FIM matrix, i.e, the CRB existence condition.

Theorem 1. The nonsingular condition of the FIM is

$$\text{rank}(\mathbf{W}) = L + |\mathbb{S}|. \tag{27}$$

The proof of Theorem 1 is given in Appendix B.

Theorem 2. Assume $\text{rank}(\mathbf{W}) = L + |\mathbb{S}|$. Then, $[\mathbf{\Gamma}^{\text{H}}(\mathbf{I} - \mathbf{D}(\mathbf{D}^{\text{H}}\mathbf{D})^{-1}\mathbf{D}^{\text{H}})\mathbf{\Gamma}]$ is positive-definite if and only if

$$\text{rank}\begin{bmatrix}\mathbf{A}_w & \mathbf{W}\end{bmatrix} = 2L + |\mathbb{S}|. \tag{28}$$

The proof of Theorem 2 is given in Appendix C.

The nonsingular condition given in (27) and (28) indicates that there are $|\mathbb{S}|$ variables to characterize the noise power corresponding to the zeroth lag in the virtual array, and it is straightforward to verify that the matrix $\mathbf{I}_{|\mathbb{S}|} \odot \mathbf{I}_{|\mathbb{S}|}$ is full column rank. As a result, to keep $\mathbf{W}$ to be full column rank, the number of signals should be less than $(|\mathbb{D}| - 1)/2$. That is, we could utilize the $(|\mathbb{D}| - 1)/2$ lags in the difference coarray to perform DOA estimation. As a result, to guarantee the CRB existing condition, the maximum number of signals must not exceed $(|\mathbb{D}| - 1)/2$. According to [23], there are $3MN + M - N$ unique lags for the coprime array configuration discussed in this paper. Therefore, considering the nonsingular condition of the CRB, the maximum number of sources that could be resolved is $(3MN + M - N)/2$.

V. Simulation Results

Consider a coprime array consisting of 10 omnidirectional antenna sensors with two coprime integers $M = 3$ and $N = 5$. The sensor positions at the two subarrays are respectively $[0, 3, 6, 9, 12]d$ and $[0, 5, 10, 15, 20, 25]d$. To verify that the coprime array can estimate more source DOAs than the number of physical sensors, we consider 15 narrowband far-field signals uniformly distributed between -65^{o} and 75^{o}. The noise vector is modeled as an independent zero-mean Gaussian random vector whose covariance matrix is $\mathbf{Q} = \text{diag}\{11.2, 5.2, 10, 3.4, 7.8, 9.2, 7, 6, 12.5, 9\}$.

To verify the effectiveness and superiority of the proposed algorithms, the results of the proposed algorithms are compared to those obtained from methods described in [12], [19] and [24]. The input signal-to-noise ratio (SNR) is defined as

$$\text{SNR} = \frac{\sigma_s^2}{2M+N-1} \sum_{l=1}^{2M+N-1} \frac{1}{\sigma_l^2} \tag{29}$$

where σ_s^2 and σ_l^2 are signal power and the noise power evaluated at the lth sensor, respectively. For Proposed Methods 1 and 2, the regularization parameter of the Lasso is set to 0.85 and 0.95, respectively, and the sample grid is uniformly distributed between $-90°$ and $90°$ with a $0.2°$ increment between adjacent grid points.

A. *Detection performance*

Fig. 2 compared the estimated spatial spectra, where the input SNR is 1 dB, and 4000 snapshots are used. From this figure, it is evident that the proposed algorithms can identify all of the sources and have sharp peaks, verifying the effectiveness of DOA estimation with a higher number of DOFs than the number of physical sensors. The method in [24] does not eliminate the effect of nonuniform noise, suffering from performance degradation. Moreover, the estimator developed in [19] only utilizes the contiguous virtual sensors, i.e., it does not take the full advantages offered by all the lags in the difference coarray. The method in [12] deletes the diagonal elements of covariance matrix, resulting in performance loss. By using the CS and the matrix completion techniques, the proposed methods utilize as many lags as possible offered in virtual array and hence achieve improved performance compared with method in [12], [19] and [24].

B. *Angular resolution ability*

We examine the angular resolution performance by considering a scenario with two closely located sources at $0.8°$ and $1.8°$. 4000 snapshots are used, and the input SNR is 1 dB. The spatial spectra estimated using the proposed algorithms and the method being compared [12], [19] and [24] are shown in Fig. 3. For clarify, we only show an angular region between $-3°$ and $5°$. It can be seen from Fig. 3 that the method in [12], [19] and [24] fails to accurately estimate the DOAs because of their close angular separation. However, all the proposed algorithms provide unbiased DOA estimates.

C. Estimation accuracy

To evaluate the estimation accuracy of the DOAs, we use the root mean square error (RMSE) as the performance indicator, which is defined by

$$\mathrm{RMSE} = \sqrt{\frac{1}{LT}\sum_{l=1}^{L}\sum_{t=1}^{T}\left[\tilde{\theta}_l(t) - \theta_l\right]^2}, \tag{30}$$

where $\tilde{\theta}_l(t)$ denotes the estimated θ_l from the tth Monte Carlo trial, $t = 1, 2, \cdots, T$. For comparison, the CRB is also included in the figure.

The RMSE performance is shown in Fig. 4 with respect to the input SNR and in Fig. 5 with respect to the number of snapshots. In Fig. 4, 4000 snapshots are used, whereas in Fig. 5, the input SNR is set to -5dB. 11 incident signals uniformly distribute between -75^{o} and 25^{o}, and results from 300 Monte Carlo trials are averaged to obtain each simulation point. Other simulation settings are identical to those used in Fig. 2. From Fig. 4, it is observed that the proposed algorithms achieve much lower RMSE than the methods being compared [12], [19] and [24]. That is due to the fact that the proposed algorithms can effectively mitigate the influence of nonuniform noise in covariance matrix and take advantage of as many lags as possible in virtual array. It is noted that, as pointed out in [39], when the number of sources is higher than that of the physical sensors, the RMSE cannot reach the CRB, which is consistent with the simulation results. In Fig. 5, it is observed that, compared to the method developed in [12], [19] and [24], the proposed methods not only offer lower RMSE results, but also achieve a faster convergence.

VI. Conclusion

Based on a coprime array, three novel DOA estimation algorithms in the presence of unknown nonuniform noise were proposed in this paper. For Proposed Method 1, by reconstructing the sample covariance matrix of physical sensors into Hermitian matrix with equal diagonal elements, the effect of unknown nonuniform noise is suppressed well. For Proposed Method 2 and Method 3, the matrix completion technique is used to recover the noise-free low-rank covariance matrix of virtual array and the entries corresponding to missing holes are filled. In addition, the proposed three algorithms take advantage of more lags than compared method, they have a better performance for DOA estimation. Simulation results demonstrate that, compared with existing method, the proposed algorithms achieve a superior performance in terms of detection capacity, angular resolution, and estimation accuracy.

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Appendix A

Considering a stochastic model with parameter set defined as $\boldsymbol{\omega} = [\tilde{\boldsymbol{\theta}}, \sigma_1^2, ...\sigma_L^2, \Psi_1^2, ..., \Psi_{|\mathbb{S}|}^2]^{\mathrm{T}}$, where $\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_1, \tilde{\theta}_2, ..., \tilde{\theta}_L] = [(d/\lambda)\text{sin}\theta_1, (d/\lambda)\text{sin}\theta_2, ..., (d/\lambda)\text{sin}\theta_L]$, whereas $\sigma_{q_1}^2, q_1 = 1, 2, ..., L$, and $\Psi_{q_2}^2, q_2 = 1, 2, ..., |\mathbb{S}|$, are the signal power of each source and the noise power of each sensor, respectively. Denote $\mathbf{R}_{\mathbb{S}} = \sum_{j=1}^{L} \sigma_j^2 \mathbf{a}(\theta_j)\mathbf{a}^{\mathrm{H}}(\theta_j) + \sum_{j=1}^{|\mathbb{S}|} \Psi_j^2 \mathbf{e}_j \mathbf{e}_j^{\mathrm{T}}$ as the covariance matrix corresponding to the physical array, where $\mathbf{e}_j$ is the jth column of the identity matrix $\mathbf{I}_{|\mathbb{S}|}$. Then, the (p, q)th entry of the FIM $\boldsymbol{\Phi}$ is given by

$$\boldsymbol{\Phi}_{p,q} = K\text{trace}\left(\mathbf{R}_{\mathbb{S}}^{-1}\frac{\partial \mathbf{R}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_p}\mathbf{R}_{\mathbb{S}}^{-1}\frac{\partial \mathbf{R}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_q}\right). \tag{A.1}$$

Since $\text{trace}(\mathbf{XYAB}) = \text{vec}^{\mathrm{H}}(\mathbf{Y}^{\mathrm{H}})(\mathbf{X}^{\mathrm{T}} \otimes \mathbf{A})\text{vec}(\mathbf{B})$, and $(\mathbf{X} \otimes \mathbf{Y})^{-1} = \mathbf{X}^{-1} \otimes \mathbf{Y}^{-1}$ when $\mathbf{X}$ and $\mathbf{Y}$ are nonsingular [41], (A.1) can be reformulated as

$$\begin{aligned}
\boldsymbol{\Phi}_{p,q} =& K\left[\text{vec}(\frac{\partial \mathbf{R}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_p})\right]^{\mathrm{H}} (\mathbf{R}_{\mathbb{S}}^{-\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}}^{-1})\text{vec}\left(\frac{\partial \mathbf{R}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_q}\right) \\
=& K\left[(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\text{vec}\left(\frac{\partial \mathbf{R}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_p}\right)\right]^{\mathrm{H}} \cdot \left[(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\text{vec}\left(\frac{\partial \mathbf{R}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_q}\right)\right] \\
=& K\left[(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\left(\frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_p}\right)\right]^{\mathrm{H}} \cdot \left[(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\left(\frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial [\boldsymbol{\omega}]_q}\right)\right],
\end{aligned} \tag{A.2}$$

where $\mathbf{r}_{\mathbb{S}} = \text{vec}(\mathbf{R}_{\mathbb{S}})$. Therefore, the FIM $\boldsymbol{\Phi}$ is expressed as

$$\boldsymbol{\Phi} = K\begin{bmatrix} \boldsymbol{\Gamma}^{\mathrm{H}}\boldsymbol{\Gamma} & \boldsymbol{\Gamma}^{\mathrm{H}}\mathbf{D} \\ \mathbf{D}^{\mathrm{H}}\boldsymbol{\Gamma} & \mathbf{D}^{\mathrm{H}}\mathbf{D} \end{bmatrix}, \tag{A.3}$$

where $\boldsymbol{\Gamma}$ is an $|\mathbb{S}|^2 \times L$ matrix given as

$$\begin{aligned}
\boldsymbol{\Gamma} =& (\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\left[\frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial \tilde{\theta}_1}, \cdots, \frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial \tilde{\theta}_L}\right] \\
=& (\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\left[\mathbf{A}_{\mathbb{S}}^* \otimes \frac{\partial \mathbf{A}_{\mathbb{S}}}{\partial \tilde{\boldsymbol{\theta}}} + \frac{\partial \mathbf{A}_{\mathbb{S}}^*}{\partial \tilde{\boldsymbol{\theta}}} \otimes \mathbf{A}_{\mathbb{S}}\right]\mathbf{P} \\
=& \jmath 2\pi(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\{\mathbf{A}_{\mathbb{S}}^* \odot [(\text{diag}(\mathbb{S})\mathbf{A}_{\mathbb{S}})] + [\text{diag}(\mathbb{S})\mathbf{A}_{\mathbb{S}}^*] \odot \mathbf{A}_{\mathbb{S}}\}\mathbf{P} \\
=& \jmath 2\pi(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\mathbf{A}_w\mathbf{P},
\end{aligned} \tag{A.4}$$

$\mathbf{P} = \text{diag}(\sigma_1^2, ..., \sigma_L^2)$, $\mathbf{A}_w = \{\mathbf{A}_{\mathbb{S}}^* \odot [\text{diag}(\mathbb{S})\mathbf{A}_{\mathbb{S}}] + [\text{diag}(\mathbb{S})\mathbf{A}_{\mathbb{S}}^*] \odot \mathbf{A}_{\mathbb{S}}\}$, and the $|\mathbb{S}|^2 \times (L + |\mathbb{S}|)$-dimensional matrix $\mathbf{D}$ is given as

$$\begin{aligned}\mathbf{D} =& (\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}} \left[\frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial \sigma_1^2}, \cdots, \frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial \sigma_L^2}, \frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial \Psi_1^2}, \cdots, \frac{\partial \mathbf{r}_{\mathbb{S}}}{\partial \Psi_{\mathbb{S}}^2}\right] \\ =& (\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}} [\mathbf{A}_{\mathbb{S}}^* \odot \mathbf{A}_{\mathbb{S}}, \mathbf{e}_1 \otimes \mathbf{e}_1, ..., \mathbf{e}_{|\mathbb{S}|} \otimes \mathbf{e}_{|\mathbb{S}|}] \\ =& (\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}} [\mathbf{A}_{\mathbb{S}}^* \odot \mathbf{A}_{\mathbb{S}}, \mathbf{I}_{\mathbb{S}} \odot \mathbf{I}_{\mathbb{S}}] \\ =& (\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}} \mathbf{W},\end{aligned} \tag{A.5}$$

where $\mathbf{W} = [\mathbf{A}_{\mathbb{S}}^* \odot \mathbf{A}_{\mathbb{S}},\ \mathbf{I}_{|\mathbb{S}|} \odot \mathbf{I}_{|\mathbb{S}|}]$. If the FIM is nonsingular, according to (A.3), the CRB of the DOAs becomes

$$\text{CRB}(\boldsymbol{\theta}) = \frac{1}{K}\{\boldsymbol{\Gamma}^{\mathrm{H}}[\mathbf{I} - \mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}]\boldsymbol{\Gamma}\}^{-1}. \tag{A.6}$$

APPENDIX B

Proof. **(Sufficient)** According to (A.5), $\mathbf{D}^{\mathrm{H}}\mathbf{D} = \mathbf{W}^{\mathrm{H}}(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-1}\mathbf{W}$. Since $\text{rank}(\mathbf{W}) = L + |\mathbb{S}|$ and $(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-1}$ is positive-definite, $\mathbf{D}^{\mathrm{H}}\mathbf{D}$ is positive-definite.

(Necessity) If $\text{rank}(\mathbf{W}) < L + |\mathbb{S}|$, there exists a non-zero vector satisfying $\mathbf{W}\mathbf{u} = \mathbf{0}$. As a result, the following equation is valid: $\mathbf{u}^{\mathrm{H}}(\mathbf{D}^{\mathrm{H}}\mathbf{D})\mathbf{u} = (\mathbf{W}\mathbf{u})^{\mathrm{H}}(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-1}(\mathbf{W}\mathbf{u}) = \mathbf{0}$, which means that $\mathbf{D}^{\mathrm{H}}\mathbf{D}$ is not positive-definite. □

APPENDIX C

Proof. **(Sufficiency)** Since $[\mathbf{I} - \mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}]$ is Hermitian and idempotent, there exists a vector $\mathbf{u}_1$ such that

$$\frac{1}{4\pi^2}\mathbf{u}_1^{\mathrm{H}}\{\boldsymbol{\Gamma}^{\mathrm{H}}[\mathbf{I} - \mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}]\boldsymbol{\Gamma}\}\mathbf{u}_1 = ||[\mathbf{I} - \mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}](\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\mathbf{A}_w\mathbf{P}\mathbf{u}_1||_2^2 \geq 0. \tag{C.1}$$

The equality is valid only if $(\mathbf{I} - \mathbf{D}(\mathbf{D}^{H}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}})(\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\mathbf{A}_w\mathbf{P}\mathbf{u}_1 = \mathbf{0}$, and there exists a vector $\mathbf{v}$ such that

$$\mathbf{A}_w\mathbf{P}\mathbf{u}_1 - \mathbf{W}\mathbf{v} = [\mathbf{A}_w \quad \mathbf{W}]\begin{bmatrix}\mathbf{P}\mathbf{u}_1 \\ -\mathbf{v}\end{bmatrix} = \mathbf{0}, \tag{C.2}$$

if $\text{rank}[\mathbf{A}_w \ \ \mathbf{W}] = 2L + |\mathbb{S}|$, then $\mathbf{P}\mathbf{u}_1 = \mathbf{0}$, implying $\mathbf{u}_1 = \mathbf{0}$. Therefore, $\{\boldsymbol{\Gamma}^{\mathrm{H}}[\mathbf{I} - \mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}]\boldsymbol{\Gamma}\}$ is positive-definite.

(Necessity) If $\text{rank}[\mathbf{A}_w \ \ \mathbf{W}] < 2L + |\mathbb{S}|$, then there exist vectors $\mathbf{x}$ and $\mathbf{y}$ such as

$$[\mathbf{A}_w \quad \mathbf{W}]\begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix} = \mathbf{0}, \tag{C.3}$$

which means

$$\mathbf{A}_w\mathbf{P}(\mathbf{P}^{-1}\mathbf{x}) = \mathbf{W}(-\mathbf{y}). \tag{C.4}$$

If $\mathbf{x} \neq \mathbf{0}$, then $[\mathbf{I} - \mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}](\mathbf{R}_{\mathbb{S}}^{\mathrm{T}} \otimes \mathbf{R}_{\mathbb{S}})^{-\frac{1}{2}}\mathbf{A}_w\mathbf{P}^{-1}\mathbf{x} = \mathbf{0}$, and $\{\mathbf{\Gamma}^{\mathrm{H}}[\mathbf{I} - \mathbf{D}(\mathbf{D}^{\mathrm{H}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{H}}]\mathbf{\Gamma}\}$ is not positive-definite. If $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{W}\mathbf{y} = \mathbf{0}$, which contradicts with $\mathrm{rank}(\mathbf{W}) = L + |\mathbb{S}|$. □

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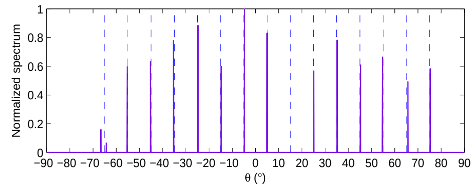
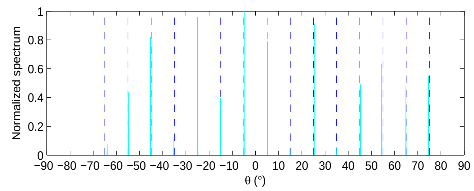
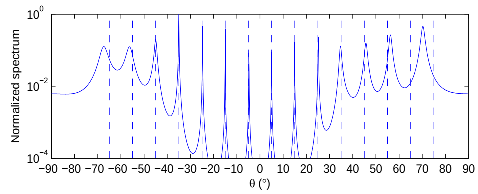


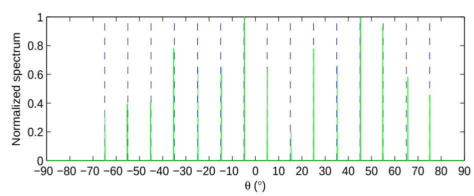
(a) DOA estimator [24]



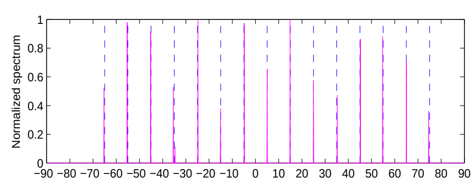
(b) DOA estimator [12]



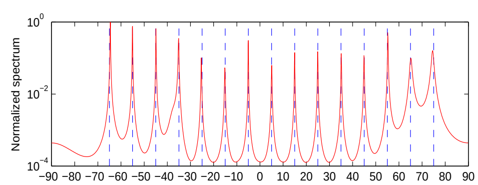
(c) DOA estimator [19]



(d) Proposed Method 1

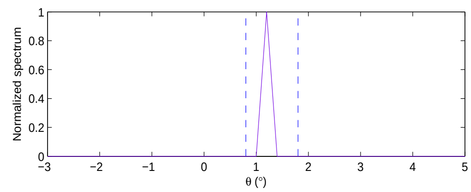


(e) Proposed Method 2

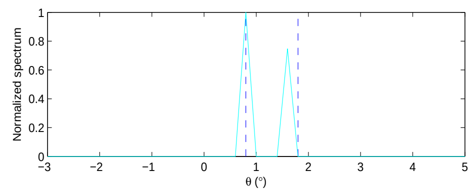


(f) Proposed Method 3

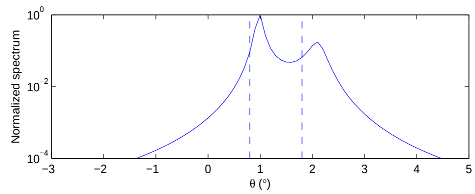
Fig. 2: Comparison of spatial spectrum in nonuniform noise environment. (1 dB input SNR and 4000 snapshots)



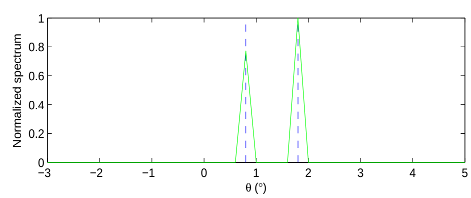
(a) DOA estimator [24]



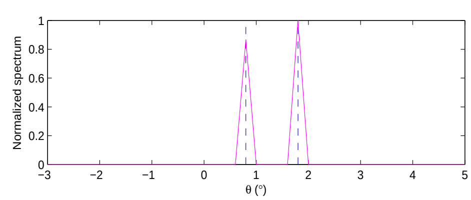
(b) DOA estimator [12]



(c) DOA estimator [19]



(d) Proposed Method 1



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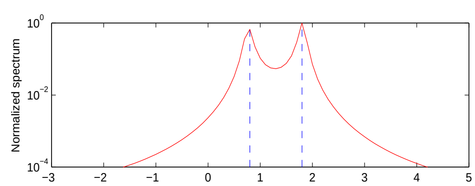

(f) Proposed Method 3

Fig. 3: Comparison of spatial spectrum in nonuniform noise environment. (1 dB input SNR and 1000 snapshots)

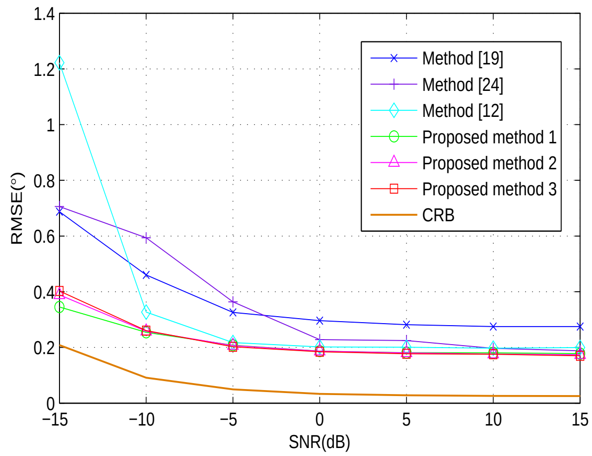

Fig. 4: RMSE vs. SNR.

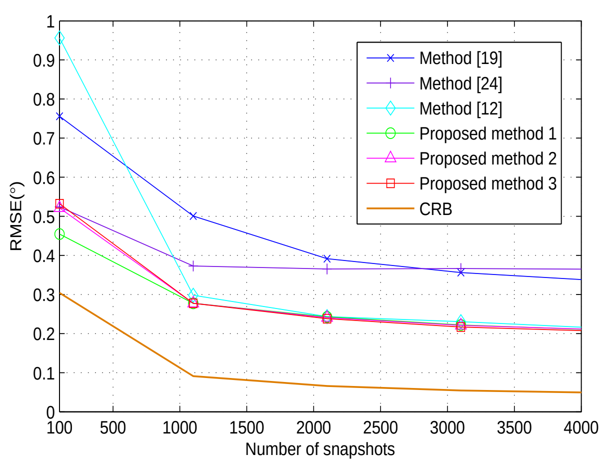

Fig. 5: RMSE vs. number of snapshots.