Off-Grid Direction-of-Arrival Estimation Using Coprime Array Interpolation

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***Abstract*—In this letter, we propose a coprime array interpolation approach to provide an off-grid direction-of-arrival (DOA) estimation. Through array interpolation, a uniform linear array (ULA) with the same aperture is generated from the deterministic non-uniform coprime array. Taking the observed correlations calculated from the signals received at the coprime array, a gridless convex optimization problem is formulated to recover all the rows and columns of the unknown correlation matrix entries corresponding to the interpolated sensors. The optimized Hermitian positive semidefinite Toeplitz matrix functions as the covariance matrix of the interpolated ULA, which enables to resolve off-grid sources. Simulation results demonstrate that the proposed array interpolation-based DOA estimation algorithm achieves improved performance as compared to existing coarray-based DOA estimation algorithms in terms of the number of achievable degrees-of-freedom and estimation accuracy.**

***Keywords*— Array interpolation, coprime array, direction-of-arrival estimation, matrix recovery, off-grid.**

I. Introduction

Direction-of-arrival (DOA) estimation is one of the fundamental techniques in the field of array signal processing, and has been successfully applied in radar, sonar, acoustics, speech, and wireless communications [1–6]. Due to the Nyquist sampling constraint, the uniform linear array (ULA) is the most commonly used array configuration. Nevertheless, the high redundancy in the ULA motivates sparse array designs for redundancy reduction. Recently, a systematically designed sparse array called the coprime array has attracted tremendous attentions [7–13]. Compared to the ULA, the coprime array offers a larger aperture and an increased number of degrees-of-freedom (DOFs), indicating a promising performance for DOA estimation.

The mainstream scheme of existing DOA estimation algorithms using the coprime array is to derive an augmented virtual array and operate the corresponding virtual array signals for DOA retrieval [14–19]. However, since the coprime array is a partially augmentable array, there exist holes in its difference coarray, rendering the derived virtual array discontiguous. To apply ULA-based DOA estimation methods in the coarray domain, a common solution is to extract the maximum contiguous segment for the subsequent coarray signal processing, such as the spatial smoothing MUSIC (SS-MUSIC) [14, 15] and the covariance matrix sparse reconstruction [16]. Clearly, there is an inherent performance loss after discarding the discontiguous virtual array sensors.

Although the sparse signal reconstruction (SSR) algorithm [18] uses the entire discontiguous virtual array, the predefined spatial sampling grids in the sparsity-based optimization problem raise concerns for angular resolution, since the actual sources are rarely located on the exact grids despite their density. In view of this, several gridless approaches are proposed to address the basis mismatch problem in the context of coarray signal processing, where the DOAs are continuously represented in the formulated optimization problems [20–23]. More recently, coarray interpolation has been carried out by interpolating additional sensors to the discontiguous virtual array for the generation of a contiguous one [24–26]. By optimizing the corresponding covariance matrix with either matrix completion or gridless reconstruction, off-grid DOA estimation can be realized with full utilization of the discontiguous virtual array.

In this letter, a novel coprime array interpolation-based DOA estimation algorithm is presented. Different from the coarray interpolation-based approaches [24–26] where the unknown correlations of the interpolated virtual ULA are located at the diagonals of the covariance matrix and correspond to the discontiguous virtual sensors, the covariance matrix of the ULA interpolated from the coprime array is directly augmented from its sample covariance matrix. As such, a number of rows and columns corresponding to the interpolated sensors are unknown, thereby preventing the utilization of matrix completion [27]. To address this issue, a gridless optimization problem is formulated to recover the covariance matrix of the interpolated ULA, where the low-rank Hermitian positive semidefinite (PSD) Toeplitz structure is incorporated as *a priori*. The optimized matrix functions as the covariance matrix of the interpolated ULA, and the ULA-based DOA estimation methods can thus be implemented to estimate off-grid sources. As a beneficial result, the recovery of the unknown correlations provides additional DOFs and, as a result, the proposed algorithm achieves improved estimation accuracy as compared to the coarray-based methods.

II. Coprime Array Signal Model

The coprime array is a union of a pair of sparse ULAs, whose sensors are respectively located at $\{0, Md, 2Md, \cdots,$

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$(N-1)Md\}$ and $\{0, Nd, 2Nd, \cdots, (M-1)Nd\}$. Here, M and N are coprime integers, and d equals to a half-wavelength, i.e., $d = \lambda/2$. Due to the coprimality, the coprime array

$$\mathbb{S} = \{Mnd \,|\, 0 \leq n \leq N-1\} \cup \{Nmd \,|\, 0 \leq m \leq M-1\} \quad (1)$$

is a non-uniform linear array containing $|\mathbb{S}| = M + N - 1$ sensors with an aperture of $\max((N-1)Md, (M-1)Nd)$, where $|\cdot|$ denotes the cardinality of a set.

Assume J uncorrelated narrowband signals impinging on the coprime array from the directions $\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_J]^{\mathrm{T}}$, where $[\cdot]^{\mathrm{T}}$ denotes the transpose. Then, the received signal vector at time index k can be modeled as

$$\boldsymbol{x}_{\mathbb{S}}(k) = \sum_{j=1}^{J} \boldsymbol{a}_{\mathbb{S}}(\theta_j) s_j(k) + \boldsymbol{n}_{\mathbb{S}}(k) = \boldsymbol{A}_{\mathbb{S}} \boldsymbol{s}(k) + \boldsymbol{n}_{\mathbb{S}}(k), \quad (2)$$

where $\boldsymbol{A}_{\mathbb{S}} = [\boldsymbol{a}_{\mathbb{S}}(\theta_1), \boldsymbol{a}_{\mathbb{S}}(\theta_2), \cdots, \boldsymbol{a}_{\mathbb{S}}(\theta_J)] \in \mathbb{C}^{|\mathbb{S}| \times J}$ is the coprime array manifold matrix with the j-th column

$$\boldsymbol{a}_{\mathbb{S}}(\theta_j) = \left[1, e^{-\jmath \frac{2\pi}{\lambda} p_2 \sin(\theta_j)}, \cdots, e^{-\jmath \frac{2\pi}{\lambda} p_{|\mathbb{S}|} \sin(\theta_j)}\right]^{\mathrm{T}} \quad (3)$$

representing the steering vector corresponding to the j-th source θ_j, $\boldsymbol{s}(k) = [s_1(k), s_2(k), \cdots, s_J(k)]^{\mathrm{T}}$ denotes the signal waveform vector, and $\boldsymbol{n}_{\mathbb{S}}(k) \sim \mathcal{CN}(\boldsymbol{0}, \sigma_n^2 \boldsymbol{I})$ denotes the complex-valued zero-mean Gaussian white noise. Here, $p_i \in \mathbb{S}, i = 1, 2, \cdots, |\mathbb{S}|$, denotes the position of the i-th sensor in the coprime array with $p_1 = 0$, $\jmath = \sqrt{-1}$ denotes the imaginary unit, σ_n^2 denotes the noise power, and $\boldsymbol{I}$ is the identity matrix with an appropriate dimension.

III. Proposed DOA Estimation Algorithm

In this section, a novel off-grid DOA estimation algorithm is proposed based on coprime array interpolation. The concept of array interpolation is introduced to generate a ULA with the same aperture as the non-uniform coprime array, where the unknown correlations corresponding to the interpolated sensors are subsequently recovered via a gridless convex optimization problem. The resulting covariance matrix corresponding to the interpolated ULA enables to estimate more off-grid sources than the number of physical sensors.

A. Array Interpolation and Statistics Initialization

The coprime array offers a systematic array configuration for sparse sensing. Nevertheless, its non-uniformity limits the adoption of the conventional ULA-based DOA estimation methods. Towards this end, as illustrated in Fig. 1, the idea of array interpolation is implemented by interpolating additional sensors into integer multiples of half-wavelength in the coprime array $\mathbb{S}$ that the physical sensors do not exist. The resulting interpolated ULA

$$\mathbb{U} = \{\ell d \,|\, 0 \leq \ell d \leq \max(\mathbb{S}), \ell \in \mathbb{Z}\} \quad (4)$$

has the same aperture as the coprime array, where the indices $\ell \in \mathbb{Z}$ are consecutive integers, and $\mathbb{S} \subset \mathbb{U}$.

It should be noted that the interpolated sensors $\mathbb{U} \backslash \mathbb{S}$, which are represented by the hollow circles in Fig. 1(b), exist in a *mathematical sense* rather than in a physical existence, indicating that the corresponding received signals are practically

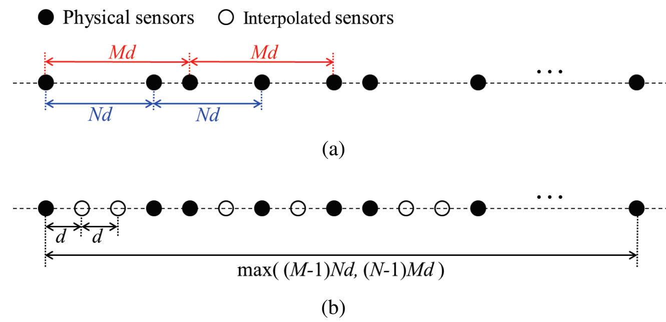


Fig. 1. Illustration of array configurations. (a) The non-uniform coprime array; (b) The interpolated ULA.

unknown. Hence, the received signals of the interpolated ULA can be initialized by augmenting $\boldsymbol{x}_{\mathbb{S}}(k)$ as

$$\langle \boldsymbol{y}_{\mathbb{U}}(k) \rangle_\ell = \begin{cases} \langle \boldsymbol{x}_{\mathbb{S}}(k) \rangle_\ell, & \ell d \in \mathbb{S}, \\ 0, & \ell d \in \mathbb{U} \backslash \mathbb{S}, \end{cases} \quad (5)$$

where $\langle \cdot \rangle_\ell$ denotes the element corresponding to the sensor located at ℓd. Accordingly, we define a $|\mathbb{U}|$-dimensional binary vector $\boldsymbol{b}$ to distinguish the sensors in the interpolated ULA $\mathbb{U}$, whose elements are 1 for the physical sensors and 0 for the interpolated sensors, i.e.,

$$\langle \boldsymbol{b} \rangle_\ell = \begin{cases} 1, & \ell d \in \mathbb{S}, \\ 0, & \ell d \in \mathbb{U} \backslash \mathbb{S}. \end{cases} \quad (6)$$

Obviously, the binary vector $\boldsymbol{b}$ is determined as long as the coprime array $\mathbb{S}$ is deployed.

The initialized received signals of the interpolated ULA $\boldsymbol{y}_{\mathbb{U}}(k)$ can be related to its theoretical version $\boldsymbol{x}_{\mathbb{U}}(k)$, whose received signals are from $|\mathbb{U}|$ sensors in $\mathbb{U}$, by

$$\boldsymbol{y}_{\mathbb{U}}(k) = \boldsymbol{x}_{\mathbb{U}}(k) \circ \boldsymbol{b}, \quad (7)$$

where $\circ$ denotes the Hadamard product operator. Here, the theoretical received signals of the interpolated ULA can be modeled as

$$\boldsymbol{x}_{\mathbb{U}}(k) = \sum_{j=1}^{J} \boldsymbol{a}_{\mathbb{U}}(\theta_j) s_j(k) + \boldsymbol{n}_{\mathbb{U}}(k) = \boldsymbol{A}_{\mathbb{U}} \boldsymbol{s}(k) + \boldsymbol{n}_{\mathbb{U}}(k), \quad (8)$$

where

$$\boldsymbol{a}_{\mathbb{U}}(\theta_j) = \left[1, e^{-\jmath \frac{2\pi}{\lambda} u_2 \sin(\theta_j)}, \cdots, e^{-\jmath \frac{2\pi}{\lambda} u_{|\mathbb{U}|} \sin(\theta_j)}\right]^{\mathrm{T}} \quad (9)$$

is the steering vector of the interpolated ULA corresponding to the j-th source θ_j, and $\boldsymbol{A}_{\mathbb{U}} = [\boldsymbol{a}_{\mathbb{U}}(\theta_1), \boldsymbol{a}_{\mathbb{U}}(\theta_2), \cdots, \boldsymbol{a}_{\mathbb{U}}(\theta_J)] \in \mathbb{C}^{|\mathbb{U}| \times J}$. Here, $u_v \in \mathbb{U}, v = 1, 2, \cdots, |\mathbb{U}|$, denotes the position of the v-th sensor in the interpolated ULA with $u_1 = 0$, and $\boldsymbol{n}_{\mathbb{U}}(k) \in \mathbb{C}^{|\mathbb{U}|} \sim \mathcal{CN}(\boldsymbol{0}, \sigma_n^2 \boldsymbol{I})$ is the additive white Gaussian noise vector corresponding to the $|\mathbb{U}|$ sensors.

Based on the initialized received signal vector of the interpolated ULA $\boldsymbol{y}_{\mathbb{U}}(k)$, its sample covariance matrix can be calculated as

$$\hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}} = \frac{1}{K} \sum_{k=1}^{K} \boldsymbol{y}_{\mathbb{U}}(k) \boldsymbol{y}_{\mathbb{U}}^{\mathrm{H}}(k), \quad (10)$$

where K is the number of snapshots, and $(\cdot)^{\mathrm{H}}$ denotes the Hermitian transpose. The correlations in the rows and columns corresponding to the interpolated sensors are zeros, while the non-zero elements in $\hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}$ are the correlations of the actual signals $\boldsymbol{x}_{\mathbb{S}}(k)$ received in the coprime array. In order to perform ULA-based DOA estimation, it is necessary to recover the unknown correlations in $\hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}$ due to the initialized zero elements in $\boldsymbol{y}_{\mathbb{U}}(k)$, such that the recovered covariance matrix approaches its theoretical version

$$\boldsymbol{R}_{\boldsymbol{x}_{\mathbb{U}}\boldsymbol{x}_{\mathbb{U}}} = \mathrm{E}\left[\boldsymbol{x}_{\mathbb{U}}(k)\boldsymbol{x}_{\mathbb{U}}^{\mathrm{H}}(k)\right] = \boldsymbol{A}_{\mathbb{U}}\boldsymbol{\Sigma}\boldsymbol{A}_{\mathbb{U}}^{\mathrm{H}} + \sigma_n^2\boldsymbol{I}. \tag{11}$$

Here, $\mathrm{E}[\cdot]$ denotes the statistical expectation operator, and $\boldsymbol{\Sigma} = \mathrm{diag}\left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_J^2\right)$ with $\sigma_j^2 = \mathrm{E}\left[|s_j(k)|^2\right]$ contains the power of J incident sources on its diagonal.

B. Covariance Matrix Recovery for Off-Grid DOA Estimation

A major limitation of the popular matrix completion technique is that it fails to recover the desired covariance matrix $\boldsymbol{R}_{\boldsymbol{x}_{\mathbb{U}}\boldsymbol{x}_{\mathbb{U}}}$ from $\hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}$, since a number of rows and columns are entirely missing in the latter [27]. Encouragingly, the theoretical covariance matrix corresponding to uncorrelated signal received at the ULA has a Hermitian Toeplitz structure [28]. This property can be utilized as the *a priori* to perform structured matrix recovery. On the other hand, the noise-free covariance matrix also exhibits a low-rank property attributed to the relatively few incident sources compared to the number of sensors in the interpolated ULA. Based on these facts, the covariance matrix of the interpolated ULA can be recovered by solving the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{z}} \quad & \mathrm{rank}\left(\mathcal{T}(\boldsymbol{z})\right) \\ \text{subject to} \quad & \left\|(\mathcal{T}(\boldsymbol{z}) - \hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}) \circ \boldsymbol{B}\right\|_F^2 \leq \delta, \\ & \mathcal{T}(\boldsymbol{z}) \succeq \boldsymbol{0}, \end{aligned} \tag{12}$$

where

$$\boldsymbol{B} = \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} \tag{13}$$

is a $|\mathbb{U}| \times |\mathbb{U}|$ dimensional binary matrix distinguishing the known (non-zero) elements and the unknown (zero) elements in the initialized sample covariance matrix $\hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}$, $\mathcal{T}(\boldsymbol{z}) \succeq \boldsymbol{0}$ denotes a Hermitian PSD Toeplitz matrix with $\boldsymbol{z} \in \mathbb{C}^{|\mathbb{U}|}$ as its first column, $\mathrm{rank}(\cdot)$ denotes the rank of a matrix, $\|\cdot\|_F$ denotes the Frobenius norm, and δ is a user-defined parameter to constrain the fitting error.

In the above optimization problem, we try to recover a low-rank Hermitian PSD Toeplitz covariance matrix of the interpolated ULA, while minimizing the difference between the known correlations in the sample covariance matrix $\hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}$ and the corresponding elements in the recovered covariance matrix $\mathcal{T}(\boldsymbol{z})$. By introducing the nuclear norm convex relaxation, the NP-hard rank minimization problem (12) can be reformulated as

$$\begin{aligned} \min_{\boldsymbol{z}} \quad & \left\|(\mathcal{T}(\boldsymbol{z}) - \hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}) \circ \boldsymbol{B}\right\|_F^2 + \xi\left\|\mathcal{T}(\boldsymbol{z})\right\|_* \\ \text{subject to} \quad & \mathcal{T}(\boldsymbol{z}) \succeq \boldsymbol{0}, \end{aligned} \tag{14}$$

where $\|\cdot\|_*$ denotes the nuclear norm of a matrix, and ξ is a regularization parameter to balance the fitting error and the nuclear norm. Since

$$\left\|\mathcal{T}(\boldsymbol{z})\right\|_* = \mathrm{Tr}\left(\sqrt{\mathcal{T}^{\mathrm{H}}(\boldsymbol{z})\mathcal{T}(\boldsymbol{z})}\right) \tag{15}$$

with $\mathrm{Tr}(\cdot)$ denoting the trace operator, the PSD constraint on $\mathcal{T}(\boldsymbol{z})$ enables to equivalently transform the nuclear norm minimization problem (14) to a trace minimization problem as

$$\begin{aligned} \min_{\boldsymbol{z}} \quad & \left\|(\mathcal{T}(\boldsymbol{z}) - \hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}) \circ \boldsymbol{B}\right\|_F^2 + \xi\,\mathrm{Tr}\left(\mathcal{T}(\boldsymbol{z})\right) \\ \text{subject to} \quad & \mathcal{T}(\boldsymbol{z}) \succeq \boldsymbol{0}. \end{aligned} \tag{16}$$

The optimization problem (16) is convex, and can be efficiently solved via interior point methods.

The optimized covariance matrix $\mathcal{T}(\hat{\boldsymbol{z}}) \in \mathbb{C}^{|\mathbb{U}|\times|\mathbb{U}|}$ is an estimate of $\boldsymbol{R}_{\boldsymbol{x}_{\mathbb{U}}\boldsymbol{x}_{\mathbb{U}}}$, i.e., the covariance matrix of the signals received by the interpolated ULA, where the incorporation of the ULA-based methods increases the number of achievable DOFs from $|\mathbb{S}|$ to $|\mathbb{U}|$. On the other hand, the matrix $\mathcal{T}(\hat{\boldsymbol{z}})$ recovered in a gridless manner enables to effectively estimate off-grid sources. The candidate ULA-based methods for DOA estimation include the subspace-based methods, such as MUSIC [29], root-MUSIC [30], ESPRIT [31], and a series of sparsity-based methods [32–34]. Considering the trade-off between estimation performance and computational complexity, we prefer to adopt the root-MUSIC to provide an off-grid DOA estimation, whose estimation accuracy is not limited by the spectrum searching process.

It should be mentioned that, while the proposed array interpolation-based DOA estimation algorithm is formulated on the systematically designed coprime array configurations, it can be readily extended to incorporate other sparse arrays, and the differences are reflected on the binary matrix $\boldsymbol{B}$ and subsequently the sample covariance matrix $\hat{\boldsymbol{R}}_{\boldsymbol{y}_{\mathbb{U}}\boldsymbol{y}_{\mathbb{U}}}$.

IV. Simulation Results

In our simulations, the pair of coprime integers are selected to be $M = 3$ and $N = 5$ for the coprime array deployment. Accordingly, we have $|\mathbb{S}| = 7$ and $|\mathbb{U}| = 13$, respectively. The regularization parameter ξ is set to 0.25.

The DOFs performance of the proposed array interpolation-based DOA estimation algorithm is shown in Fig. 2 by depicting the its spatial spectra based on $\mathcal{T}(\hat{\boldsymbol{z}})$, where the incident sources are assumed to be uniformly distributed in

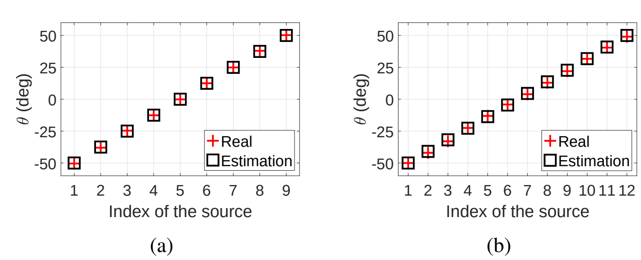
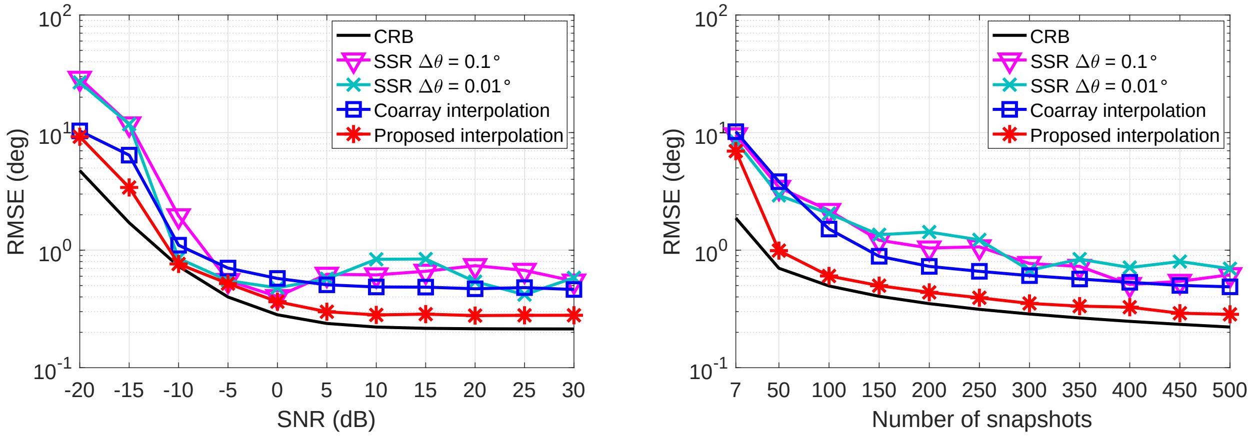


Fig. 2. DOFs illustration of the proposed DOA estimation algorithm. (a) $J = 9$; (b) $J = 12$.



(a) (b)

Fig. 3. Comparison of DOA estimation performance. (a) RMSE versus SNR when $K = 500$; (b) RMSE versus number of snapshots when SNR = 10 dB.

$[-50°, 50°]$ with SNR = 0 dB and $K = 500$. For the case that the number of sources is $J = 9$, which exceeds the number of physical sensors as well as the number of maximum achievable DOFs for the coarray-based SS-MUSIC algorithm [14], it can be seen from Fig. 2(a) that the proposed algorithm effectively resolves all the nine sources. When the number of sources increases to $J = 12$, the number of maximum achievable DOFs for the subspace-based methods based on $\mathcal{T}(\hat{z})$, the spatial spectrum shown in Fig. 2(b) indicates that the proposed algorithm is still effective. Recall the fact that only $|\mathbb{S}| = 7$ physical sensors are utilized for receiving the signals, it is clear that additional DOFs are obtained by the proposed array interpolation-based DOA estimation algorithm.

The DOA estimation performance of the proposed algorithm is compared to two existing coarray-based DOA estimation algorithms, i.e., the SSR algorithm [18] and the gridless reconstruction-based coarray interpolation algorithm [25, 26]. Consider nine equal-power sources impinging on the deployed coprime array from the directions uniformly distributed in $[-52.78°, 50.35°]$, where the number of off-grid sources exceeds the number of physical sensors. For a fair comparison, both the gridless reconstruction-based coarray interpolation algorithm and the proposed coprime array interpolation-based DOA estimation algorithm apply the root-MUSIC method to the optimized covariance matrix for DOA estimation. The predefined spatial sampling grid interval $\Delta\theta$ for the SSR algorithm is selected to be $0.1°$ and $0.01°$. The performance is compared in terms of the root mean square error (RMSE), and $L = 500$ Monte-Carlo trials are performed for each data point (either SNR or number of snapshots). The RMSE is defined as

$$\text{RMSE} = \sqrt{\frac{1}{LJ}\sum_{l=1}^{L}\sum_{j=1}^{J}\left(\hat{\theta}_{j,l} - \theta_j\right)^2}, \tag{17}$$

where $\hat{\theta}_{j,l}$ denotes the estimate of the j-th source θ_j in the l-th Monte-Carlo trial. Meanwhile, the Cramér-Rao bound (CRB) [35], whose Fisher information matrix remains nonsingular in this scenario, is also plotted for reference.

The RMSE versus the SNR is compared in Fig. 3(a) where the number of snapshots is $K = 500$. It is shown in Fig. 3(a) that the performance of the proposed coprime array interpolation-based algorithm outperforms the gridless reconstruction-based coarray interpolation algorithm, and its performance trend is consistent with the CRB predictions. For the SSR algorithm, peaks of the irregular spurious spatial spectra introduced by the sparsity-based approaches account for the fluctuant trend when the SNR is higher than 0 dB. Such performance trend does not improve when a denser grid with $\Delta\theta = 0.01°$ is used in the underlying multiple off-grid sources case. When the number of snapshots varies with SNR = 10 dB, as shown in Fig. 3(b), the proposed algorithm provides the best estimation accuracy in coping with off-grid sources, and its RMSE is close to the CRB when the number of snapshots is larger than 50.

It is worth mentioning that the coarray-based algorithms exploiting the contiguous part of the virtual array, such as the SS-MUSIC algorithm [14] and the covariance matrix sparse reconstruction algorithm [16], fail to perform effective DOA estimation in this scenario, since the contiguous coarray ranges only between -7 and 7, thus indicating that the maximum number of resolvable sources is 7. In contrast, the proposed algorithm is capable of resolving all nine off-grid sources with its higher number of the DOFs.

V. Conclusions

We have proposed a novel off-grid DOA estimation algorithm using coprime array interpolation in this letter. The additional sensors are interpolated to convert the non-uniform coprime array to a ULA. Based on the initialized statistics, a structured matrix recovery problem is formulated to recover the unknown correlations corresponding to the interpolated sensors in a gridless manner. It is demonstrated that additional DOFs can be obtained with the recovered covariance matrix corresponding to the interpolated ULA, and the proposed algorithm achieves better estimation accuracy for off-grid sources than the existing algorithms operating coarray signals.

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