Non-Overlapping Identical Division of Integer Set

Ajay N. Namboothiri, Md Waqeeb T. S. Chowdhury, and Yimin D. Zhang

Department of Electrical and Computer Engineering, Temple University, Philadelphia, PA 19122, USA

***Abstract*—This paper develops a systematic approach to identically dividing a consecutive integer set into two or more non-overlapping subsets, where each subset generates consecutive difference lags with reduced redundancy. Unlike the identical nested partitioning scheme which assigns every element in the integer set to a subset, the proposed identical division scheme allows some integers to remain unassigned. This flexibility enables more effective subset formation with reduced redundancy. A general expression for selecting subset elements is devised to facilitate systematic design, and the resulting difference lags are analyzed. Compared to identical nested partitioning schemes, the proposed division method achieves higher difference lags for the same number of elements used in each subset. As an example of practical applications, the proposed identical division scheme is employed to efficiently design sparse step-frequency radar waveforms for interference-free multi-radar operation, with its effectiveness demonstrated through target range estimation.**

***Index Terms*—identical division, sparse radar waveform, multi-radar system, interference-free waveform design.**

I. Introduction

Robust and efficient design of sparse sensor arrays [1–11] and sparse radar waveforms [12–18] has garnered significant attention in recent years due to their ability to enable simple, low-cost, and low-complexity hardware implementations. Most existing sparse arrays and waveforms are designed with a single platform or user in mind and, therefore, cannot be directly adapted to effectively support multiple platforms. In practice, it is highly desirable to efficiently utilize the space, time, and frequency resources derived from equally spaced array elements and uniformly sampled waveforms through shared usage across multiple platforms. This is especially important in the modern era, where the wireless spectrum is increasingly congested. Toward this end, a novel partitioning method was recently proposed in [19] for interference-free multi-platform operation. This method partitions a consecutive integer set is partitioned into two or more identical, non-overlapping subsets, each generating consecutive difference lags. These subsets can then be allocated to multiple distinct platforms, enabling the shared use of resources, such as sensors, step frequencies, and slow-time pulses, across platforms without mutual interference.

Many signal processing applications assume uniform Nyquist sampling in one or more domains, resulting in one- or multi-dimensional consecutive discrete sets. On the other hand, recent developments of difference coarray and sparsity-based processing techniques have enabled efficient and reliable direction-of-arrival (DOA) estimation of signals using sparse arrays [3, 11, 20–22]. Similarly, with the recent advances in sparse step frequencies and sparse slow-time pulses [14–16, 18], it becomes possible to estimate target Doppler and range using only a subset of these consecutive discrete samples. Identical partitioning of such consecutive sets into multiple non-overlapping subsets allows multiple radars to coexist and share time and/or frequency resources without interference, maintaining similar performance. An example of this concept is the design of non-uniform pulse repetition interval (PRI) radars or sparse step-frequency radars. In such designs, sparsity along the slow-time or step frequency can be structured in a non-overlapping manner, enabling multiple radars to fully utilize the total coherent processing interval (CPI) or bandwidth simultaneously without interference. After examining three possible identical partition methods, namely, localized, interleaved, and nested partitions, it was observed in [19] that the nested partitioning of integer sets provided the largest set of unique and consecutive difference lags. Additionally, closed-form expressions were provided for the non-overlapping subsets and their difference lags for identical partitioning of one- and multi-dimensional consecutive integer sets.

In this paper, we develop a systematic approach to identically dividing a consecutive integer set into two or more non-overlapping subsets. The proposed identical division method is designed so that each subset is identical and, for a given number of elements per subset, achieves higher consecutive difference lags compared to the nested partition method introduced in [19]. Consequently, the proposed method outperforms nested partitioning in terms of difference lags for the same subset size. This identical division of integer sets provides valuable insights for designing sparse antenna arrays and radar waveforms that deliver superior performance compared to those based on nested partitioning. The effectiveness of the proposed approach is demonstrated through simulation results presented in Section IV.

Notations: We denote lower-case bold characters as vectors, and upper-case blackboard bold characters as sets. $|\mathbb{M}|$ represents the cardinality of set $\mathbb{M}$, whereas $\otimes$ and $\cup$ respectively denote the Kronecker product and union operators.

II. Two-Subset Identical Division

Consider an M-element consecutive integer set $\mathbb{Q} = \{1, \cdots, M\}$. The goal is to divide $\mathbb{Q}$ into two non-overlapping subsets, $\mathbb{Q}_1$ and $\mathbb{Q}_2$, with the remaining integers, which are not assigned to any subset, denoted as $\mathbb{Q}_p$, i.e., $\mathbb{Q} = \mathbb{Q}_1 \cup \mathbb{Q}_2 \cup \mathbb{Q}_p$. Subsets $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are designed to have identical, rotated patterns and, therefore, produce the same consecutive difference lags.

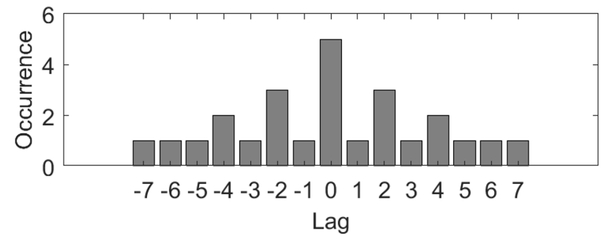
In the following, we provide a brief overview of the identical nested partitioning method proposed in [19] and outline the general rule and closed-form expressions for the two-subset

This work was supported in part by the National Science Foundation (NSF) under Grant ECCS-2236023 and the Air Force Office of Scientific Research (AFOSR) under Grant FA9550-23-1-0255.

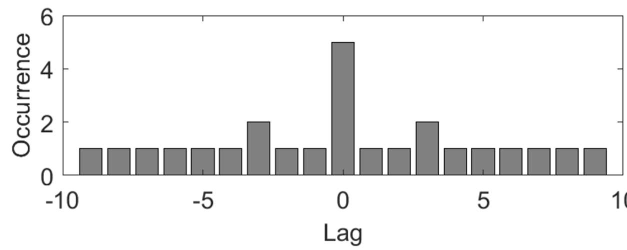
(a) Identical nested partition ($N = 3$)

(b) Identical division ($K = 1$ and $N = 3$)

Fig. 1: Two-subset nested partitioning and identical division of integer sets with 5 elements in each subset.



(a) Identical nested partitioning



(b) Identical division

Fig. 2: Lag occurrences for two-subset identical nested partitioning and identical division with 5 elements in each subset.

identical division. Through analytical evaluation, we demonstrate that the proposed identical division approach achieves higher consecutive and unique difference lags compared to nested partition for the same number of elements in subsets $\mathbb{Q}_1$ and $\mathbb{Q}_2$.

A. Review of Two-Subset Identical Nested Partition

When $M = 2N + 4$ with $N \geq 1$, as shown in Fig. 1(a) for the $N = 3$ case, the two-subset nested partitioning has two elements as the inner group, and N elements in the outer group, each separated by two. The total number of elements in both subsets is $|\mathbb{Q}_1| = |\mathbb{Q}_2| = N + 2$. The two subsets are respectively given as

$$\mathbb{Q}_1 = \{1, 2, 4, 6, \cdots, 2N, 2N + 2\} \tag{1}$$

and

$$\mathbb{Q}_2 = \{3, 5, \cdots, 2N - 1, 2N + 1, 2N + 3, 2N + 4\}. \tag{2}$$

The two subsets have the same consecutive difference lags between $-2N-1$ and $2N+1$. For the partition pattern shown in Fig. 1(a), each subset produces consecutive lags between $-2N - 1 = -7$ and $2N + 1 = 7$, and the occurrence weights of the difference lags are shown in Fig. 2(a).

B. Two-Subset Identical Division

We now consider dividing the consecutive integer set $\mathbb{Q}$ into two identical subsets, $\mathbb{Q}_1$ and $\mathbb{Q}_2$, with the flexibility that not all elements must be assigned to a subset. We first develop an approach where the core elements in a subset are separated by three, which will be referred to as "core separation of 3." In this configuration, elements 1 and 3 form the inner group of subset $\mathbb{Q}_1$, along with N outer group elements. Both subsets contain $N + 2$ elements and are defined as

$$\mathbb{Q}_1 = \{1, 3, 6, 9, \cdots, 3N - 3, 3N, 3N + 1\} \tag{3}$$

and

$$\mathbb{Q}_2 = \{4, 5, 8, 11, \cdots, 3N - 1, 3N + 2, 3N + 4\}. \tag{4}$$

In each subset, in addition to the elements in the core segment, located in the middle, the elements at both ends of the subset form lag-1 and lag-2 pairs, respectively, ensuring that all lags are consecutive.

To match the number of subset elements shown in Fig. 1(a), the two-subset identical division of an integer set is depicted in Fig. 1(b). It is observed that, a few elements, shown in gray, are not assigned to either subset $\mathbb{Q}_1$ or $\mathbb{Q}_2$. For identical division with N outer groups, the location of these unallocated integers are given by

$$\mathbb{Q}_p = \{2, 7, 12 \cdots, 3N - 2, 3N + 3\}. \tag{5}$$

It is noted that, for a two-subset identical division, $M = 3N + 4$, where $N \geq 2$. We define the following $(3N+4)\times 1$ masking vectors for subsets $\mathbb{Q}_1$ and $\mathbb{Q}_2$ as

$$\boldsymbol{b}_g(n) = \begin{cases} 1, & \text{if } n \in \mathbb{Q}_g, \\ 0, & \text{if } n \notin \mathbb{Q}_g, \end{cases} \tag{6}$$

where $g = 1, 2$.

The difference lag set for the gth subset, $g = 1, 2$, is denoted as

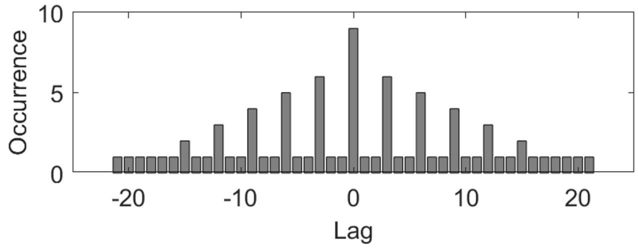
$$\mathbb{D}_g = \{D \mid D = u - v, u \in \mathbb{Q}_g, v \in \mathbb{Q}_g\}. \tag{7}$$

It can be inferred from (3) and (4) that subsets $\mathbb{Q}_1$ and $\mathbb{Q}_2$ have the same consecutive difference lags between $-3N$ and $3N$. Compared to nested partitioning, which achieves consecutive difference lags between $-2N - 1$ and $2N + 1$, it can be concluded that, for the same number of outer groups or, equivalently, the same number of elements in each subset, the identical division method provides a greater number of consecutive difference lags.

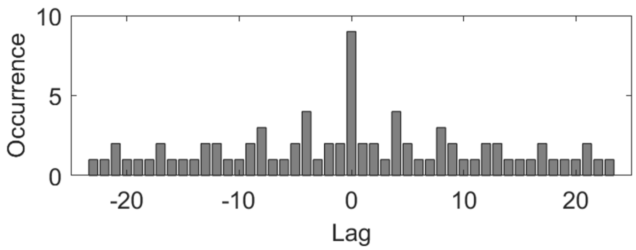
For $N = 3$ outer groups, as shown in Fig. 2(b), the identical division method provides consecutive difference lags ranging from -9 to 9. Furthermore, since the identical division method achieves a greater number of consecutive lags with the same number of subset elements, it produces difference lags with fewer redundancies compared to the identical nested partitioning method.

C. Generalization of Two-Subset Identical Division

The two-subset identical division approach can also utilize other core separations. When a subset contains a large number



(a) Core separation of 3



(b) Core separation of 4

Fig. 3: Lag occurrences for identical division with different core element separations in subsets containing 9 elements.

of elements, adopting a larger separation becomes advantageous. For instance, with $\tilde{N}+4$ elements in a subset, the following "core separation of 4" configuration,

$$\tilde{\mathbb{Q}}_1 = \{1, 2, 3, 7, \cdots, 4\tilde{N}-1, 4\tilde{N}+2, 4\tilde{N}+4\} \tag{8}$$

and

$$\tilde{\mathbb{Q}}_2 = \{4, 6, 9, 13, \cdots, 4\tilde{N}+5, 4\tilde{N}+6, 4\tilde{N}+7\}, \tag{9}$$

produces consecutive lags between $-4\tilde{N}-3$ and $4\tilde{N}+3$. When a subset contains more than 7 elements, this configuration generates more consecutive lags than the "core separation of 3" counterpart.

Fig. 3 compares the lag occurrences for these two configurations in subsets contains 9 elements each. With a core separation of 4, the configuration yields consecutive lags between -23 and 23, whereas core separation of 3 results in lags between -21 and 21.

In the sequel, we focus exclusively on the results for subsets with a core separation of 3. However, extending the results to other configurations, such as those with a core separation of 4, is straightforward.

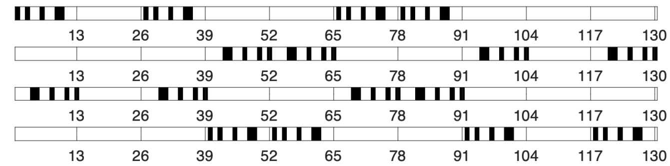
III. Multi-Subset Identical Division

In this section, we extend the concept of two-subset identical division to $G = 2^g$ identical subsets for $g \geq 1$ such that $\mathbb{Q} = \mathbb{Q}_1 \cup \mathbb{Q}_2 \cup \cdots \cup \mathbb{Q}_G \cup \mathbb{Q}_p$.

Let us consider $g = 2$, allowing $\mathbb{Q}$ to be divided into four non-overlapping identical subsets. The masking vectors for the four subsets can be obtained as

$$\boldsymbol{b}_{g_1,g_2} = \boldsymbol{b}_{g_2} \otimes \boldsymbol{b}_{g_1}, \tag{10}$$

where $g_1 \in \{1,2\}$ and $g_2 \in \{1,2\}$. Let N_1 and N_2 denote the total number of outer group elements for $\mathbb{Q}_1$ and $\mathbb{Q}_2$, respectively, where $N_1 \geq 2$ and $N_2 \geq 2$ may take different values. It is observed in (10) that the dimension of the masking vector $\boldsymbol{b}_{g_1,g_2}$ is $(3N_1+4)(3N_2+4)\times 1$, and the number of elements in each of the four subsets is $(N_1+2)(N_2+2)$. The resulting difference lags for each subset obtained by identical division are consecutive between $-(3N_2+4)(3N_1)-(3N_2)$ and $(3N_2+4)(3N_1)+(3N_2)$, whereas that for a subset obtained by identical nested partitioning is between $-(2N_2+4)(2N_1+1)-(2N_2+1)$ and $(2N_2+4)(2N_1+1)+(2N_2+1)$ [19].



(a) Four subset identical division of a consecutive integer set.

(b) Union set of the four identically divided subsets

Fig. 4: Four-subset identical division of a 130-element consecutive integer set.

We now illustrate a four-subset identical division of a consecutive integer set, where $N_1 = 3$ and $N_2 = 2$. This allows us to identically divide a 130-element consecutive integer set into four non-overlapping subsets, as shown in Fig. 4(a). Fig. 4(b) depicts the set obtained by combining all the four subsets, indicating that some integer elements are not allocated to any of the subsets.

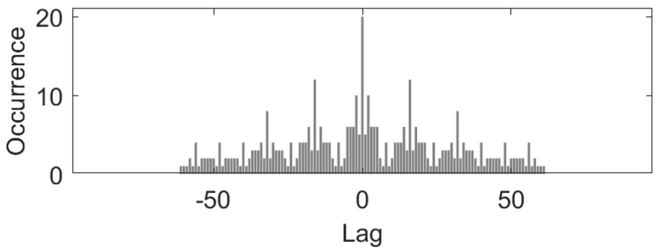
The difference lags achieved by four-subset identical nested partitioning and division for $N_1 = 3$ and $N_2 = 2$ are compared in Fig. 5. In Fig. 5(a), we observe that a subset obtained from the four-subset nested partition achieves only 61 consecutive difference lags, whereas the proposed identical division method achieves 96 lags. Moreover, it can be clearly observed in Fig. 5 that the identical division method results in significantly lower redundancies in the difference lags.

Fig. 6 compares the number of non-negative difference lags achieved by each subset for both the identical nested partitioning and division methods. It is observed that the identical division method consistently produces more difference lags, and the gap between the two methods increases as the number of elements in each subset grows.

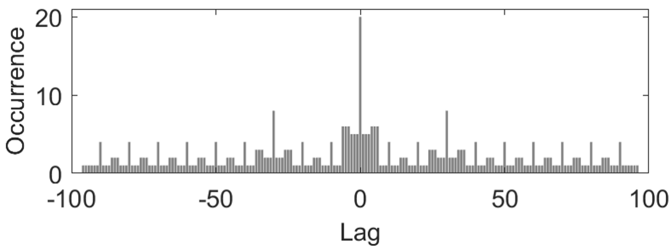
IV. Simulation Results

In this section, we design a sparse step-frequency radar waveform by allocating the available step frequencies across the entire bandwidth into four non-overlapping subsets. Each subset is used to design an orthogonal radar waveform for each user. We then compare the range profile of the radar waveforms designed using both the identical nested partition and divisions, with the same number of step frequencies being used in each case.

In a step-frequency radar, the available bandwidth B is equally divided into N_f step frequencies with a step size



(a) Difference lags from nested partitioning ($N_1 = 3$ and $N_2 = 2$)



(b) Difference lags from identical division ($N_1 = 3$ and $N_2 = 2$)

Fig. 5: Comparison of the achievable difference lags between four-subset identical nested partitioning and division methods for same number of subset elements.

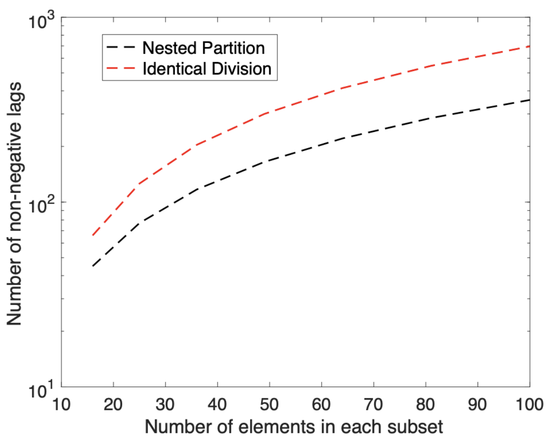
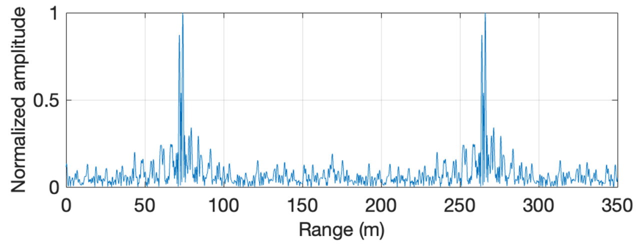
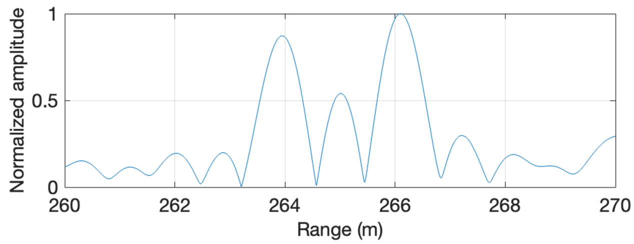


Fig. 6: Number of non-negative lags versus the number of elements in each subset.

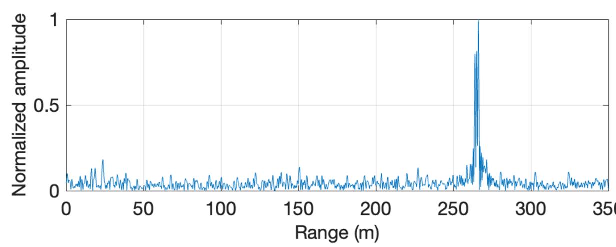


(a) Estimated range profile

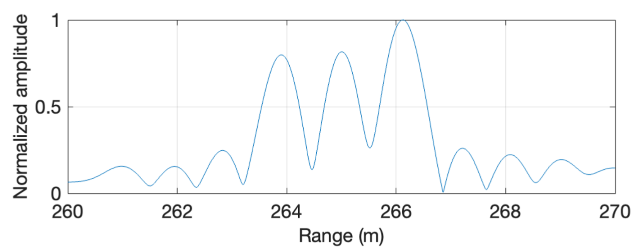


(b) Zoomed-in view

Fig. 7: Estimated range profile using identical partitioning.



(a) Estimated range profile



(b) Zoomed-in view

Fig. 8: Estimated range profile using identical division.

$f_\Delta = B/N_f$. Therefore, the nth step frequency is given as:

$$f_n = f_0 + (n-1)f_\Delta, \quad n = 1, 2, \cdots, N_f, \tag{11}$$

where f_0 is the base frequency. The maximum unambiguous range of the step-frequency radar is given as $R_{\max} = c/(2f_\Delta)$.

We consider a radar system with base frequency $f_0 = 77$ GHz, and the signal bandwidth is $B = 200$ MHz. Three targets are located at ranges of 264 m, 265 m, and 266 m, respectively, within the radar's field of view. To keep a similar number of step frequencies in each waveform, we choose $N_1 = N_2 = 6$ for both the waveforms designed using the identical nested partitioning and division methods. For identical nested partitioning, the total number of available step frequencies is $(2N_1+4)(2N_2+4) = 256$ with a corresponding step size of $\Delta f = 781.25$ kHz, resulting in a maximum unambiguous range of $R_{\max} = 192$ m. Fig. 7(a) shows the range estimates over the ranges between 0 m and 350 m, whereas Fig. 7(b) shows the zoomed-in view for the range between 260 m and 270 m. As can be seen in Fig. 7, the estimated range profiles for the three targets show alias along with the true ranges, preventing unambiguous range estimates for the targets.

On the other hand, for the identical division method, with $N_1 = N_2 = 6$, the total number of available step frequencies is $(3N_1+4)(3N_2+4) = 484$, resulting in a step size of $\Delta f = 413.22$ kHz. As a result, the maximum unambiguous range becomes $R_{\max} = 363$ m. Fig. 8 shows the successful and unambiguous detection of the three targets without any aliasing. Therefore, for the same number of step frequencies in each radar waveform, the step-frequency waveform designed using the identical division method achieves a higher maximum unambiguous range due to the higher number of consecutive difference lags, compared to the identical nested partitioning counterpart.

V. Conclusion

In this paper, we proposed an identical division method for grouping consecutive integer sets into multiple non-overlapping subsets. Both analytical results and numerical examples demonstrated that the subsets formed through this identical division approach provide a higher number of consecutive difference lags compared to those obtained using the identical nested partitioning scheme proposed in [19]. The advantages of using this identical division method for efficient resource allocation in multi-platform scenarios has been highlighted through an example of sparse step-frequency radar waveform design. The radar waveform designed using the identical division method achieves a significantly higher maximum unambiguous range compared to its identical nested partitioning counterpart, while using the same number of step frequencies per waveform.

VI. References

[1] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[2] P. P. Vaidyanathan, and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.

[3] S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Process.*, vol. 63, no. 6, pp. 1377–1390, March 2015.

[4] C. L. Liu and P. P. Vaidyanathan, "Super nested arrays: Linear sparse arrays with reduced mutual coupling–Part I: Fundamentals," *IEEE Trans. Signal Process.*, vol. 64, no. 15, pp. 3997–4012, Aug. 2016.

[5] C. L. Liu and P. P. Vaidyanathan, "Super nested arrays: Linear sparse arrays with reduced mutual coupling–Part II: High-order extensions," *IEEE Trans. Signal Process.*, vol. 64, no. 16, pp. 4203–4217, Aug. 2016.

[6] A. Ahmed, Y. D. Zhang, and B. Himed, "Effective nested array design for fourth-order cumulant-based DOA estimation," in *Proc. IEEE Radar Conf.*, Seattle, WA, May 2017, pp. 998–1002.

[7] J. Liu, Y. Zhang, Y. Lu, S. Ren, and S. Cao, "Augmented nested arrays with enhanced DOF and reduced mutual coupling," *IEEE Trans. Signal Process.*, vol. 65, no. 21, pp. 5549–5563, Nov. 2017.

[8] Z. Zheng, W. Wang, Y. Kong and Y. D. Zhang, "MISC Array: A new sparse array design achieving increased degrees of freedom and reduced mutual coupling effect," *IEEE Trans. Signal Process.*, vol. 67, no. 7, pp. 1728–1741, April 2019.

[9] A. Ahmed and Y. D. Zhang, "Generalized non-redundant sparse array designs," *IEEE Trans. Signal Process.*, vol. 69, pp. 4580–4594, Aug. 2021.

[10] S. Zhang, A. Ahmed, Y. D. Zhang, and S. Sun, "Enhanced DOA estimation exploiting multi-frequency sparse array," *IEEE Trans. Signal Process.*, vol. 69, pp. 5935–5946, Oct. 2021.

[11] C. Zhou, Y. Gu, Y. D. Zhang, and Z. Shi, "Sparse array interpolation for direction-of-arrival estimation," in M. G. Amin (ed.), *Sparse Arrays for Radar, Sonar, and Communications*, Wiley-IEEE Press, 2024.

[12] J. Li and Z. Chen, "Research on random PRI PD radar target velocity estimate based on NUFFT," in *Proc. IEEE CIE Int. Conf. Radar*, Chengdu, China, Oct. 2011, pp. 1801–1803.

[13] S. Na, K. V. Mishra, Y. Liu, Y. C. Eldar, and X. Wang, "TenDSuR: Tensor-based 4D sub Nyquist radar," *IEEE Signal Process. Lett.*, vol. 26, no. 2, pp. 237–241, 2018.

[14] K. V. Mishra, S. Mulleti, and Y. C. Eldar, "RaSSteR: Random sparse step-frequency radar," arXiv preprint arXiv: 2004.05720, 2020.

[15] W. P. Plessis, "Simultaneous unambiguous range and Doppler through non-uniform sampling," in *Proc. IEEE Radar Conf.*, Florence, Italy, Sept. 2020, pp. 1–6.

[16] S. Sun and Y. D. Zhang, "4D automotive radar sensing for autonomous vehicles: A sparsity-oriented approach," *IEEE Sel. Topics Signal Process.*, vol. 15, no. 4, pp. 879–891, June 2021.

[17] Z. Mao, S. Liu, Y. D. Zhang, L. Han, and Y. Huang, "Joint DoA-range estimation using space-frequency virtual difference coarray," *IEEE Trans. Signal Process.*, vol. 70, pp. 2576–2591, May 2022.

[18] L. Xu, S. Sun, K. V. Mishra, and Y. D. Zhang, "Automotive FMCW radar with difference co-chirps," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 59, no. 6, pp. 8145–8165, Dec. 2023.

[19] Y. D. Zhang and S. Sun, "Identical partitioning of consecutive integer set," in *Proc. IEEE Sensor Array and Multich. Signal Process. Workshop*, Corvallis, OR, July 2024, pp. 1–5.

[20] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the music algorithm," in *Proc. Digital Signal Process. and Signal Process. Edu. Meeting*, Sedona, AZ, 2011, pp. 289–294.

[21] C.-L. Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," *IEEE Signal Process. Lett.*, vol. 22, no. 9, pp. 1438–1442, Sept. 2015.

[22] S. Liu, Z. Mao, Y. D. Zhang, and Y. Huang, "Rank minimization-based Toeplitz reconstruction for DoA estimation using coprime array," *IEEE Commun. Lett.*, vol. 25, no. 7, pp. 2265–2269, July 2021.