Cumulant-Based Direction-of-Arrival Estimation Using Multiple Co-Prime Frequencies

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***Abstract*—In this paper, we propose a novel direction-of-arrival (DOA) estimation technique based on multiple co-prime frequencies and fourth-order statistics of the received signals. The utilization of multiple frequencies provides virtual sensors at the receiver array, thereby resulting in extended aperture, higher number of degrees-of-freedom, and greater flexibility compared to the commonly used single frequency-based methods. The set of lags achieved from the resulting virtual antenna elements is further extended by exploiting higher-order statistics-based difference co-array approach. The proposed scheme yields the fourth-order difference co-array which offers a significantly greater number of lags compared to the sparse array techniques used by existing DOA estimation methods. Simulation results verify the effectiveness of the proposed technique.**

keywords: Direction-of-arrival estimation, sparse array, fourth-order difference co-array, multiple frequencies.

I. Introduction

Direction-of-arrival (DOA) estimation is an important problem in array signal processing which finds applications in radar, sonar, wireless communications, and radio astronomy [1, 2]. While second-order statistics (SOS)-based methods are extensively used to determine the DOA of impinging electromagnetic waves, techniques based on higher-order statistics (HOS) are known to provide a higher number of degrees-of-freedom (DOFs) so that more sources than the number of array elements can be resolved [3–5]. Moreover, HOS-based approaches are more effective in suppressing Gaussian random components, such as thermal noise, because the HOS of Gaussian random variables is zero, enabling accurate parameter estimation of non-Gaussian signals [6].

Efficient under-determined DOA estimation can be achieved by using sparse sensor arrays in the context of co-arrays [7]. Particularly, for a given number of physical sensors, the minimum redundancy array (MRA) [8] provides the highest number of consecutive lags in the resulting difference co-array, whereas the minimum hole array (MHA) [9] provides the least possible number of holes in the yielding difference co-array. However, both MRA and MHA structures do not have simple closed-form analytical expressions, thus making it difficult to formulate their effective configurations for a given number of antenna elements. Recent advancements in sparse array design have resulted in array configurations, such as co-prime array (CPA) [10], which can be systematically designed and analyzed. A CPA consists of a pair of sparse linear sub-arrays such that the inter-element spacings of both sub-arrays are related to two co-prime integers and the number of elements in each sub-array is proportional to the inter-element spacing of the other sub-array. The achievable number of the DOFs and that of the consecutive and unique lags of a CPA are well studied in [11]. Utilization of sparse sensor arrays based on fourth-order cumulants (FOC) of signals has recently attracted great attention because they provide an increased number of DOFs to resolve a far higher number of sources than conventional SOS based methods [12–14].

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The concept of CPA has been recently extended to use a sparse uniform linear array (ULA) exploiting two or more frequencies which can be associated with a co-prime relationship [15, 16]. In this scheme, the ULA corresponding to each frequency acts as a sub-array in the CPA. As such, a CPA-like structure is achieved using a single physical ULA. The signals of multiple frequencies can be obtained either passively or actively. In the former, such signals are obtained at the receive array by filtering the received signals using different filters centered at the proper frequencies. For the latter, multiple-frequency signals are transmitted from a transmit antenna or an antenna array, and return signals corresponding to the respective frequencies are filtered at the receive array accordingly. The analysis of strategies and achievable number of DOFs employing multiple frequencies has been performed in [15, 16] when the virtual array is obtained based on the SOS.

In this paper, we address the problem of FOC-based DOA estimation using multiple frequencies. In this context, the maximum achievable number of DOFs and unique lags is analytically evaluated. It is observed that an FOC-based formulation for a multiple frequency approach provides a much higher number of lags and DOFs compared to existing SOS-based approaches. Simulation results verify the effectiveness of the proposed approach.

Notations: A lower (upper) case bold letter denotes a vector (matrix). Specifically, $\mathbf{I}_N$ and $\mathbf{0}_N$ denote the $N \times N$ identity and zero matrices, respectively. $(\cdot)^*$ and $(\cdot)^T$ denote the complex conjugation and transpose operations. $\mathrm{E}(\cdot)$ stands for statistical expectation.

II. Problem Formulation

A. Signal Model

Consider an L-sensor ULA, and Q narrowband independent and non-Gaussian stationary signals reflected from far-field targets are impinging on the array from angles $\theta_1, \theta_2, \ldots, \theta_Q$, respectively. The signal received at each antenna is filtered to render $K = 2$ sub-streams at corresponding frequencies f_1 and f_2. The $L \times 1$ received signal vector $\mathbf{x}_k(t)$ at the receive array corresponding to the signal frequency f_k is expressed as:

$$\mathbf{x}_k(t) = \mathbf{A}\left(f_k\right)\mathbf{s}_k(t) + \mathbf{n}_k(t), \tag{1}$$

where $\mathbf{A}\left(f_k\right)$ is the $L \times Q$ array manifold matrix corresponding to the frequency f_k, $\mathbf{s}_k(t)$ represents the $Q \times 1$ signal vector, $\mathbf{n}_k(t)$ is the $L \times 1$ white noise vector that follows the joint complex Gaussian distribution with zero mean and covariance matrix $\sigma_n^2 \mathbf{I}_L$. We assume $\mathbf{s}_k(t)$ to be random and zero-mean and, therefore, $\mathbf{x}_k(t)$ are also zero-mean random vectors.

Denote D as the inter-element spacing, and $f_0 = c/2D$ as the reference frequency for which the inter-element spacing is half wavelength, i.e., $D = \lambda_0/2$, where c is the velocity of wave propagation. The two frequencies f_1 and f_2 are chosen as integer multiple of f_0, i.e., $f_1 = M_1 f_0$ and $f_2 = M_2 f_0$ such that M_1 and M_2 are co-prime integers. We refer to M_k as the dilation factor of the inter-element spacing.

Then, each column of matrix $\mathbf{A}\left(f_k\right)$ is the steering vector corresponding to f_k and can be written as:

$$\begin{aligned}\mathbf{a}\left(\theta_q, f_k\right) &= [1, e^{-j2\pi D\sin(\theta_q)/\lambda_k}, \ldots, e^{-j2\pi D(L-1)\sin(\theta_q)/\lambda_k}]^T \\ &= [1, e^{-j\pi M_k \sin(\theta_q)}, \ldots, e^{-j\pi M_k(L-1)\sin(\theta_q)}]^T,\end{aligned} \tag{2}$$

where $\lambda_k = \lambda_0/M_k$ is the wavelength corresponding to f_k. That is, the normalized position of the lth sensor, evaluated at frequency f_k, is $(l-1)M_k$.

It is clear that the combination of the ULA and two frequencies renders two dilated sub-arrays which resemble those in a CPA. However, there are several differences to the ordinary CPA. One such difference is that, while CPA assumes different numbers of sensors in each sub-array, the same number of sensors are used in both dilated sub-arrays.

B. Fourth-Order Statistics

Denote x_z as a zero-mean random variable that is associated with a normalized sensor position of z. Then, for zero-mean random variables x_{z_1}, x_{z_2}, x_{z_3} and x_{z_4}, the FOC cumulant can be calculated by:

$$\begin{aligned}&\mathrm{cum}\{x_{z_1}, x_{z_2}^*, x_{z_3}, x_{z_4}^*\} \\ &\quad = \mathrm{E}\left\{x_{z_1}x_{z_2}^*x_{z_3}x_{z_4}^*\right\} - \mathrm{E}\left\{x_{z_1}x_{z_2}^*\right\}\mathrm{E}\left\{x_{z_3}x_{z_4}^*\right\} \\ &\qquad - \mathrm{E}\left\{x_{z_1}x_{z_4}^*\right\}\mathrm{E}\left\{x_{z_3}x_{z_2}^*\right\} - \mathrm{E}\left\{x_{z_2}^*x_{z_4}^*\right\}\mathrm{E}\left\{x_{z_1}x_{z_3}\right\},\end{aligned} \tag{3}$$

where $\mathrm{cum}(\cdot)$ is the cumulant operator. The resultant lag obtained from the above expression is $l = z_2 + z_4 - z_1 - z_3$. The different lags obtained from various combination of array sensors with their respective normalized sensor positions form a virtual array, whose sensors are the different lag positions. It is important to note that the simultaneous use of conjugated and unconjugated array data in the FOC computation yields both sum co-array and difference co-array sensors in the resultant virtual array.

Stacking the cumulants corresponding to all unique lags obtained from Eq. (3) yields a vector $\mathbf{y}$, which can be expressed in the following form:

$$\mathbf{y} = \tilde{\mathbf{A}}\mathbf{b} \tag{4}$$

where $\tilde{\mathbf{A}}$ is the extended array manifold of the resultant virtual array whose columns act as a dictionary associated with all possible impinging DOAs, and $\mathbf{b}$ is a sparse vector whose non-zero entries correspond to the cumulants of the signal corresponding to a DOA.

III. Direction-of-Arrival Estimation

Depending upon whether to use array outputs belonging to the same frequency or different frequencies, the FOC results can be classified into self-lags and cross-lags. The self-lag measurement vectors are computed from Eqs. (1) and (3) as:

$$\mathbf{y}^{(k,k,k,k)} = \tilde{\mathbf{A}}^{(k,k,k,k)}\mathbf{b}^{(k,k,k,k)}, \tag{5}$$

where the superscript (k,k,k,k) indicates that the same frequency component is used for $k = 1,2$. Similarly, cross-lag measurement vectors are computed as:

$$\mathbf{y}^{(k,m,n,o)} = \tilde{\mathbf{A}}^{(k,m,n,o)}\mathbf{b}^{(k,m,n,o)}, \tag{6}$$

where $k,m,n,o \in \{1,2\}$ are the frequency indexes to be used. In this work, we assume that the 1st and 3rd components use signals in one frequency, and the 2nd and the 4th components use signals in the other frequency. As such,

$$\mathbf{y}^{(k,m,k,m)} = \tilde{\mathbf{A}}^{(k,m,k,m)}\mathbf{b}^{(k,m,k,m)}, \tag{7}$$

where $m = 3 - k$.

After the FOC-based observation vectors are determined, we can perform DOA estimation using classical methods, such as MUSIC, or compressive sensing methods. As the cumulant vector is rank one, the MUSIC algorithm requires spatial smoothing to be applied to restore the full rank. As such, only the consecutive lags can be utilized. On the other hand, compressive sensing methods can fully utilize all the available lags, regardless of whether they are consecutive or not [17]. In addition, because the self- and cross-lags correspond to different values of $\mathbf{b}$, group sparse compressive sensing methods enable more effective fusion of these components [15, 16].

In this work, we use both self- and cross-lags, and compressive sensing based methods for DOA estimation. In particular, the complex multi-task Bayesian compressive sensing (CMT-BCS) algorithm [18, 19] is applied.

Note that, more than two mutually co-prime frequencies can also be utilized to further enhance the DOF. Unlike SOS-based virtual array formation where the SOS results are computed pair-wisely, the FOC-based approach enables simultaneous processing of up to four frequencies at the same time, resulting in a greater number of unique as well as consecutive lags. The advantages of using more than two co-prime frequencies is demonstrated with numerical results in Section V.

IV. Achievable DOFs

In this section, we derive the maximum number of DOFs and the number of unique lags of the difference co-arrays achieved by utilizing FOC and two co-prime frequencies.

For a pair of co-prime dilation factors, M_1 and M_2, the locations of the elements in the two virtual sub-arrays are

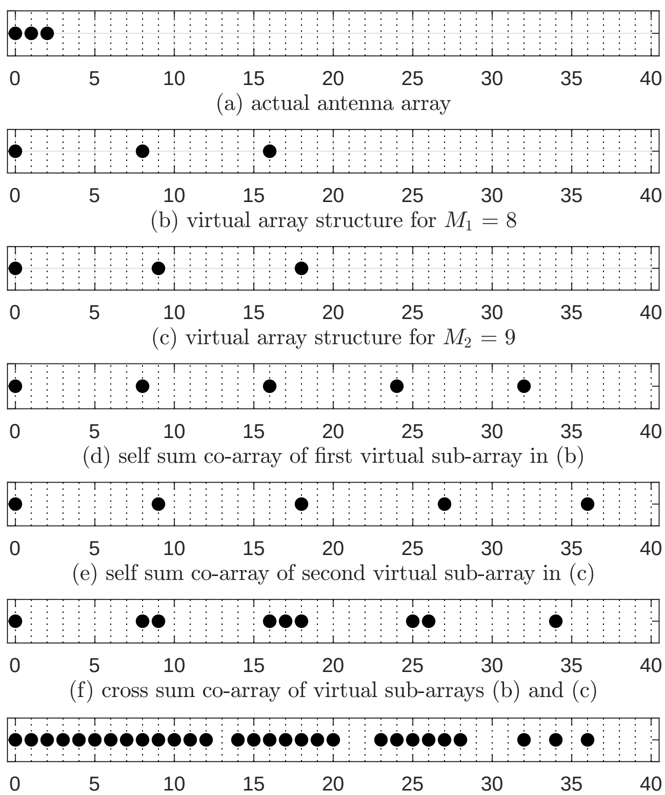


Fig. 1. A sparse ULA with two co-prime frequencies and its FOC-based difference co-array formation (only non-negative lags are shown)

respectively given by:

$$P_1 = \{M_1 l_1 d \mid 0 \leq l_1 \leq L-1\}, \quad P_2 = \{M_2 l_2 d \mid 0 \leq l_2 \leq L-1\}, \tag{8}$$

where l_1 and l_2 are integers.

Without loss of generality, we assume that the co-prime dilation factors satisfy $M_1 < M_2$. In this case, the normalized aperture of the resulting co-prime array is $M_2(L-1)$.

From Eq. (3), we can observe that the fourth-order difference co-array of the co-prime array in Eq. (8) contains both the sum and difference co-arrays of all the virtually dilated sub-arrays. As expected, a higher number of DOFs is achieved when compared to SOS-based methods.

The self sum co-arrays of Eq. (8) can be expressed as follows:

$$P_{s1} = \{M_1 l_1' d \mid 0 \leq l_1' \leq 2L-1\}, \quad P_{s2} = \{M_2 l_2' d \mid 0 \leq l_2' \leq 2L-1\}, \tag{9}$$

where l_1' and l_2' are integers. Similarly, we can express the cross sum co-array between the two virtual sub-arrays P_1 and P_2 in Eq. (8) as follows:

$$P_{s12} = \{(M_1 l_1 + M_2 l_2)d\}. \tag{10}$$

The combined sum co-array of the virtual arrays in Eq. (8) can be expressed as:

$$P_s = P_{s1} \cup P_{s2} \cup P_{s12}. \tag{11}$$

According to [14], the FOC-based virtual sensors can be calculated by evaluating the difference co-array of its sum co-arrays. We will use the same strategy to yield the FOC-based difference co-array of the sub-arrays in Eq. (8) which can be achieved by evaluating the difference co-array of Eq. (11).

We first find the lags contained in the difference co-array of sum co-arrays in Eq. (9) as:

$$L_f = L_c \cup L_c^-, \tag{12}$$

where

$$L_c = \{(M_1 l_1' - M_2 l_2')d\}, \quad L_c^- = \{(M_2 l_2' - M_1 l_1')d\}. \tag{13}$$

Here, L_c^- is the mirrored set of L_c. Note that, in Eq. (12) and (13), only P_{s1} and P_{s2} are considered for synthesizing the FOC-based lags. The result of Eq. (12) is similar to the SOS-based results discussed in [16] except that the virtual sensors in Eq. (9) is now $2L-1$ instead of L as in the SOS-based counterparts. Therefore, the number of unique lags γ in Eq. (12) can be obtained by replacing L in [16] with $2L-1$ as follows:

$$\gamma = 2(2L-1)^2 - 1 - \max\{0, 4L-3-M_2\} \min\{M_1+1, 4L-3-M_1\}. \tag{14}$$

Thus, the maximum possible unique lags from Eq. (12) is $8L^2 - 8L + 1$ which is calculated using the difference co-array of $P_{s1} \cup P_{s2}$. In order to find the maximum unique achievable number of FOC-based lags from the proposed approach, we must also include the unique lags contributed by the difference co-array of P_{s12} in our analysis. We can derive that the additional unique FOC-based lags contributed by the difference co-array of P_{s12} is $2L^2 - 4L + 2$.

The maximum possible number of unique lags γ_{max} by considering all the combinations of FOC-based sum and difference co-array lags can be summarized in the following proposition.

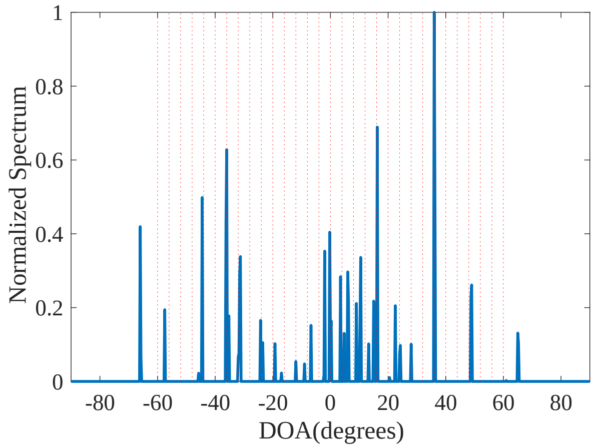
Proposition 1: For a virtual array constructed from a ULA with inter-element spacing D using two co-prime frequencies, where $D = M_1\lambda_1/2 = M_2\lambda_2/2$, the maximum possible number of unique lags, γ_{max}, achievable from the fourth-order difference co-array is given by:

$$\gamma_{max} = 10L^2 - 12L + 3. \tag{15}$$

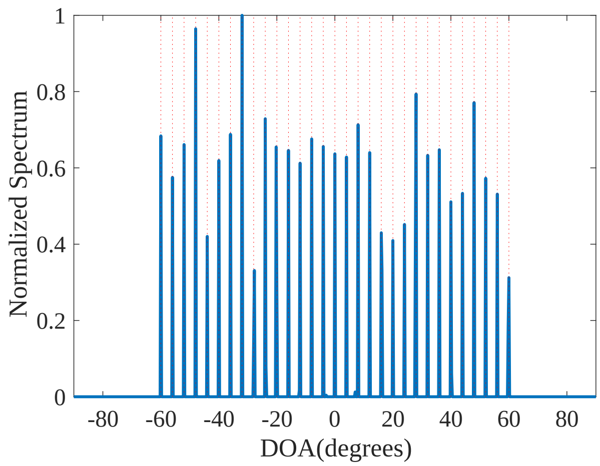
When the maximum number of lags are achieved, the number of DOFs can be determined by $(\gamma_{max} + 1)/2$.

TABLE I. NUMBER OF UNIQUE CO-ARRAY LAGS ACHIEVED BY DIFFERENT SPARSE ARRAY STRUCTURES

Method	No. of sensors	Configuration	Max. no. of unique lags
SO ULA [16]	3	$M_1 = 4, M_2 = 5$	17
Proposed FO ULA	3	$M_1 = 8, M_2 = 9$	57
SO ULA [16]	7	$M_1 = 12, M_2 = 13$	97
Proposed FO ULA	7	$M_1 = 24, M_2 = 25$	409
SO ULA [16]	9	$M_1 = 16, M_2 = 17$	161
Proposed FO ULA	9	$M_1 = 32, M_2 = 33$	705

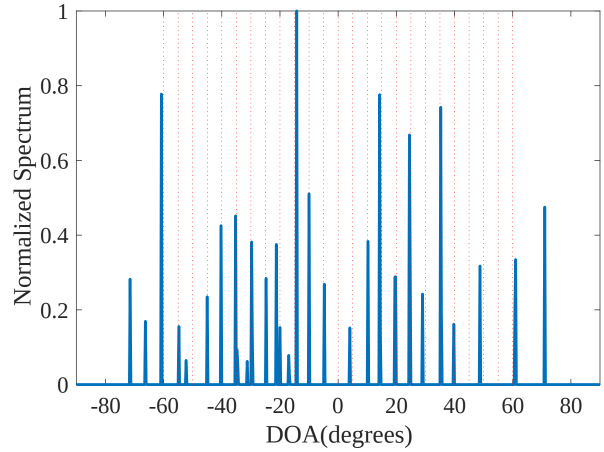


(a) SOS-based method

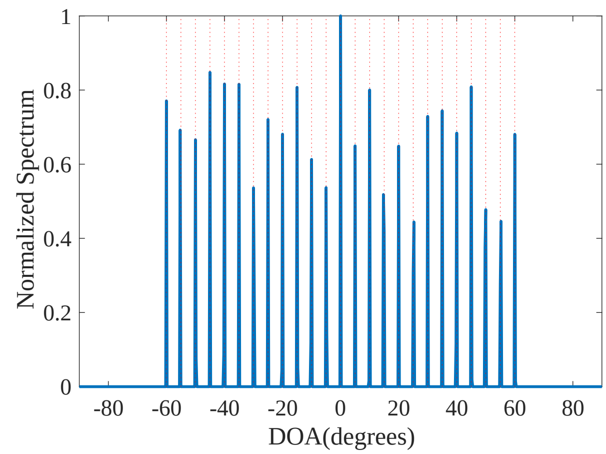


(b) Proposed FOC-based method

Fig. 2. Normalized spatial spectrum for 31 sources uniformly distributed from -60^o to 60^o (5 sensors, SNR = 0 dB, 1,000,000 snapshots, $M_1 = 16$ and $M_2 = 17$).



(a) Two-frequencies with $M_1 = 8$ and $M_2 = 9$



(b) Three frequencies with $M_1 = 8$, $M_2 = 9$, and $M_3 = 11$

Fig. 3. Normalized spatial spectrum using the proposed FOC-based method for 25 sources uniformly distributed from -60^o to 60^o (3 sensors, SNR = 0 dB, 1,000,000 snapshots).

Fig. 1 shows the formation of dilated co-prime arrays and their resulting FOC-based difference co-array for an array having 3 antenna elements and the dilation factors of $M_1 = 8$ and $M_2 = 9$.

The maximum possible number of unique lags achievable from the proposed technique is $10L^2 - 12L + 3$, which is far greater than the SOS-based counterpart which offers a maximum number of possible unique lags of $2L^2 - 1$. The comparison of the achievable number of DOFs for the proposed FOC-based approach and the existing SOS-based method is illustrated in Table I.

V. Simulation Results

In this section, we present simulation results to demonstrate the performance of the proposed FOC-based DOA estimation technique and compare it with the existing SOS-based method [16]. We consider independent sources impinging on a ULA with an input signal-to-noise radio (SNR) of 0 dB. The phase difference between the received signals corresponding to different co-prime frequencies is independently and uniformly distributed over $[0, 2\pi)$. 1,000,000 snapshots of data are considered and all the unique lags are utilized for DOA estimation.

In the first example, we have 31 sources impinging on a 5-sensor array exploiting two co-prime frequencies with respective dilation factors of $M_1 = 16$ and $M_2 = 17$. For the case of SOS-based DOA estimation, we obtain 49 unique lags which corresponds to 25 DOFs. It is observed in Fig. 2(a) that the SOS-based method exploiting the two frequencies clearly fails to resolve all the 31 sources because of the insufficient number of DOFs. For the proposed FOC-based DOA estimation method, the number of resulting unique lags increases to 193 and the corresponding number of DOFs is 97. As shown in Fig. 2(b), the proposed FOC-based technique resolves all the 31 sources. These results clearly verify that, for an array exploiting two co-prime frequencies, the proposed DOA estimation technique based on the FOC offers a significant increase of DOFs and provides substantial performance improvement as compared to the SOS-based counterpart.

In the second example, we evaluate the resolution capability of the proposed approach using different numbers of co-prime frequencies. We consider 25 sources impinging on a 3-sensor array and two cases are considered. In the first case, two co-prime frequencies are used with dilation factors $M_1 = 8$ and $M_2 = 9$, respectively. In this case, the proposed FOC-based DOA technique provides 57 unique lags. It can be observed in Fig. 3(a) that the proposed method is unable to resolve all the sources successfully. In the second case shown in Fig. 3(b), an additional co-prime frequency with dilation factor of $M_3 = 11$ is used, thus increasing the number of unique lags to 79. It is observed that the DOA estimation performance of the proposed technique is significantly improved by using more mutually co-prime frequencies.

VI. Conclusion

In this paper, we have presented a novel FOC-based DOA estimation method utilizing multiple frequencies which results in enhanced source resolution capabilities. The proposed approach provides a higher number of unique co-array lags compared to existing SOS-based methods. The number of achievable unique lags provided by the FOC-based DOA estimation techniques was derived for the case of two co-prime frequencies. Simulation results show that the proposed method outperforms existing techniques by resolving more sources with less sensors.

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