Two-Dimensional DOA Estimation Using Parallel Coprime Subarrays

Si Qin[†], Yimin D. Zhang[‡], and Moeness G. Amin[†]

[†]Center for Advanced Communications, Villanova University, Villanova, PA 19085, USA

[‡]Department of Electrical and Computer Engineering, Temple University, Philadelphia, PA 19122, USA

***Abstract*—A conventional coprime array is a linear array, which consists of two uniform linear subarrays to construct an effective difference coarray with certain desirable characteristics. In this paper, we propose a parallel coprime array structure and a novel algorithm for two-dimensional (2-D) direction-of-arrival (DOA) estimation. By vectorizing the cross-covariance matrix of subarray data, the resulting virtual difference coarray enables resolving more signals than the number of antennas. The 2-D DOA estimation problem is cast as two separate one-dimensional DOA estimation problems, where the estimated azimuth and elevation angles can be properly associated. Compared with other methods, such as, the propagator method (PM) and the rank-reduction (RARE) based algorithms, the proposed method resolves more signals and achieves improved estimation performance.**

Keywords*—*Two-dimensional (2-D) direction-of-arrival estimation, coprime array, parallel subarrays.

I. Introduction

Direction-of-arrival (DOA) estimation determines the spatial spectrum of the impinging electromagnetic waves on the antenna array. It finds variety of applications in radar, sonar, radio astronomy, and mobile communication systems [1]. In recent years, two-dimensional (2-D) DOA estimation has been rigorously investigated. Among existing techniques, the subspace-based methods, such as multiple signal classification (MUSIC) [2] and estimation of signal parameters via rotational invariance techniques (ESPRIT) [3], are most popular due to their high-resolution direction finding capabilities. However, these approaches require 2-D search and use either eigenvalue decomposition (EVD) or singular value decomposition (SVD) operation to obtain noise or signal subspaces, which is computationally costly when the number sensors is large.

In [4], a well-known propagator method (PM) was proposed to find one-dimensional (1-D) DOA with significantly reduced computational complexity. The propagator can be estimated from the covariance matrix of the received signals with a least square process and without any EVD or SVD operation. By utilizing the PM, [5]–[9] extended the 1-D DOA estimation to 2-D cases. In [5], an algorithm was developed by exploiting two separate rectangular planar subarrays. However, it still requires an exhaustive 2-D peak search. In [6], a fast algorithm was proposed based on two parallel uniform linear arrays (ULAs), which requires an additional pair matching between the 2-D azimuth and elevation angle estimation and the antennas are only partially used. In [7], an approach was proposed to achieve automatically paired 2-D DOA estimation using the same array configuration. In addition, it achieves improved estimation performance because all elements of the array were fully used. However, it has an estimation failure problem when the elevation angle is high, which are typical in mobile communication environments. To solve that problem, other techniques were proposed to use three parallel ULAs, but they do not effectively use the achievable array degree-of-freedoms (DOFs).

Generally, all the PM-based methods do not take full advantage of all sensors and the number of DOFs cannot exceed $N_t/2$, where N_t is the number of array sensors. Thus, the maximum number of estimated signals was very limited. In [10], an approach based on rank-reduction (RARE) [11] for a modified three parallel ULAs, which can be treated as subarrays displaced from a long ULA. By constructing a long virtual ULA, it makes use of more sensors and resolve up to $(N_t - 3)$ signals.

In this paper, we consider the problem of detecting more signals than the number of array sensors, which is of tremendous interest in various applications [12], [13]. Toward this purpose, a higher number of DOFs is usually achieved by exploiting a sparse array under the coarray equivalence. Among a number of techniques that are available for sparse array construction, the recent proposed coprime array is very attractive due its systematical design capability and DOF analysis [14]. The coprime array configuration, which consists of two collocated uniform linear subarrays, enables estimation of DOAs up to $\mathcal{O}(MN)$ signals when N_t is $\mathcal{O}(M+N)$ for a coprime integers M and N. Several coprime configurations were developed to achieve higher DOFs and more flexible array design [15], [16]. However, these different coprime configurations are mainly limited to the 1-D case.

In this paper, we propose a parallel coprime structure and a novel algorithm for 2-D DOA estimation using the coprime array configuration. By vectorizing the cross-covariance matrix of subarray data, the resulting virtual difference coarray enables to resolve more signals than the number of antennas. The 2-D DOA estimation problem is cast as two separate one-dimensional DOA estimation problems in the proposed technique. It does not require the EVD operation and exhaustive 2-D peak search. In addition, the estimated azimuth and elevation angles can be paired automatically. Compared with other propagator method (PM) based and rank-reduction (RARE) based algorithms, the proposed method can resolve more signals.

Notations: We use lower-case (upper-case) bold characters to denote vectors (matrices). In particular, $\mathbf{I}_N$ denotes the

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$N \times N$ identity matrix. $(.)^*$ implies complex conjugation, whereas $(.)^T$ and $(.)^H$ respectively denote the transpose and conjugate transpose of a matrix or vector. $\mathrm{vec}(\cdot)$ denotes the vectorization operator that turns a matrix into a vector by stacking all columns on top of the another, and $\mathrm{diag}(\mathbf{x})$ denotes a diagonal matrix that uses the elements of $\mathbf{x}$ as its diagonal elements. $\mathrm{E}(\cdot)$ is the statistical expectation operator and $\otimes$ denotes the Kronecker product. $\mathcal{CN}(x|a,b)$ denotes that random variable x follows a complex Gaussian distribution with mean a and variance b. $\arg(x)$ returns the phase of a complex variable x. $\mathbb{N}^+$ denotes the set of positive integers. Krank($\mathbf{\Phi}$) denotes the Kruskal rank of $\mathbf{\Phi}$, returning the largest integer r such that every r columns of $\mathbf{\Phi}$ are linearly independent.

II. Signal Model

As illustrated in Fig. 1, the array configuration consists of a coprime pair of uniform linear subarrays, i.e., a $2M$-element subarray with an interelement spacing of Nd, and an N-element subarray with an interelement spacing of Md, where $d = \lambda/2$ is the unit interelement spacing, and λ is the wavelength of the carrier frequency. The two integers $M \in \mathbb{N}^+$ and $N \in \mathbb{N}^+$, $M < N$ are chosen to be coprime, i.e., their greatest common divisor is one. The difference to the conventional coprime arrays for 1-D DOA estimation lies in the fact that these two subarrays are placed parallel with a distance $d = \lambda/2$.

Assume that Q far-field narrowband uncorrelated signals $s_q(t)$, $t = 1,\ldots,T$, for $q = 1,\ldots,Q$, impinge on the array from the angles (θ_q, ϕ_q), where θ_q and ϕ_q denote the elevation angle and azimuth angle of the qth signal, respectively. Let $\mathbf{s}(t) = [s_1(t),...,s_Q(t)]^T$. Then, the $2M \times 1$ and $N \times 1$ data vectors received at the subarrays are, respectively, expressed as

$$\mathbf{x}_1(t) = \sum_{q=1}^{Q} \mathbf{a}_1(\theta_q,\phi_q)s_q(t) + \mathbf{n}_1(t) = \mathbf{A}_1\mathbf{s}(t) + \mathbf{n}_1(t), \quad (1)$$

$$\mathbf{x}_2(t) = \sum_{q=1}^{Q} \mathbf{a}_2(\theta_q,\phi_q)s_q(t) + \mathbf{n}_2(t) = \mathbf{A}_2\mathbf{s}(t) + \mathbf{n}_2(t), \quad (2)$$

where

$$\mathbf{a}_1(\theta_q,\phi_q) = \left[1, e^{j\pi N \sin(\theta_q)\sin(\phi_q)}, \ldots, e^{j\pi(2M-1)N \sin(\theta_q)\sin(\phi_q)}\right]^T, \quad (3)$$

$$\mathbf{a}_2(\theta_q,\phi_q) = \begin{aligned}&\Big[1, e^{j\pi M \sin(\theta_q)\sin(\phi_q) + j\pi\sin(\theta_q)\cos(\phi_q)}, \ldots, \\ &\ldots, e^{j\pi(N-1)M\sin(\theta_q)\sin(\phi_q)+j\pi\sin(\theta_q)\cos(\phi_q)}\Big]^T\end{aligned} \quad (4)$$

are the steering vectors of the subarrays corresponding to angles (θ_q, ϕ_q) for $q = 1,\ldots,Q$. The $2M \times Q$ manifold of subarray 1 is $\mathbf{A}_1 = [\mathbf{a}_1(\theta_1,\phi_1),...,\mathbf{a}_1(\theta_Q,\phi_Q)]$, and the $N \times Q$ manifold of subarray 2 is $\mathbf{A}_2 = [\mathbf{a}_2(\theta_1,\phi_1),...,\mathbf{a}_2(\theta_Q,\phi_Q)]$. In addition, the elements of the noise vectors $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are assumed to be independent and identically distributed (i.i.d.) random variables following the complex Gaussian distribution $\mathcal{CN}(0, \sigma_n^2\mathbf{I}_{2M})$ and $\mathcal{CN}(0, \sigma_n^2\mathbf{I}_N)$, respectively.

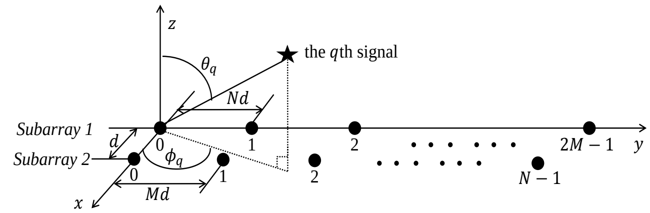

Fig. 1. Geometry of the array configuration.

III. 2-D DOA Estimation

A. The Proposed Approach

In Fig. 2, we define α_q as the angle between the qth signal and the $y-$axis, and β_q is defined as the corresponding angle between the signal and the $x-$axis, for all $q = 1,\ldots,Q$. As such, we can obtain the following relationships

$$\cos(\alpha_q) = \sin(\theta_q)\sin(\phi_q), \quad (5)$$

$$\cos(\beta_q) = \sin(\theta_q)\cos(\phi_q). \quad (6)$$

In addition, the steering vectors of two subarrays can be rewritten as

$$\begin{aligned}\mathbf{a}_1(\theta_q,\phi_q) &= \tilde{\mathbf{a}}_1(\alpha_q) \\ &= \left[1, e^{j\pi N\cos(\alpha_q)}, \ldots, e^{j\pi(2M-1)N\cos(\alpha_q)}\right]^T,\end{aligned} \quad (7)$$

$$\begin{aligned}\mathbf{a}_2(\theta_q,\phi_q) &= \tilde{\mathbf{a}}_2(\alpha_q)e^{j\pi\cos(\beta_q)} \\ &= \left[1, e^{j\pi M\cos(\alpha_q)}, \ldots, e^{j\pi(N-1)M\cos(\alpha_q)}\right]^T e^{j\pi\cos(\beta_q)}.\end{aligned} \quad (8)$$

The corresponding received vectors can be expressed as

$$\mathbf{x}_1(t) = \tilde{\mathbf{A}}_1\mathbf{s}(t) + \mathbf{n}_1(t), \quad (9)$$

$$\mathbf{x}_2(t) = \tilde{\mathbf{A}}_2\mathbf{B}\mathbf{s}(t) + \mathbf{n}_2(t), \quad (10)$$

with the manifolds $\tilde{\mathbf{A}}_1 = [\tilde{\mathbf{a}}_1(\alpha_1),...,\tilde{\mathbf{a}}_1(\alpha_Q)]$ and $\tilde{\mathbf{A}}_2 = [\tilde{\mathbf{a}}_2(\alpha_1),...,\tilde{\mathbf{a}}_2(\alpha_Q)]$. In addition,

$$\mathbf{B} = \mathrm{diag}([e^{j\pi\cos(\beta_1)}, \ldots, e^{j\pi\cos(\beta_Q)}]). \quad (11)$$

The cross-covariance matrix between the data vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ is obtained as

$$\begin{aligned}\mathbf{R}_{\mathbf{x}_{12}} &= \mathrm{E}[\mathbf{x}_1(t)\mathbf{x}_2^H(t)] = \tilde{\mathbf{A}}_1\mathbf{R}_{\mathbf{ss}}\mathbf{B}^H\tilde{\mathbf{A}}_2^H \\ &= \sum_{q=1}^{Q}\sigma_q^2 e^{-j\pi\cos(\beta_q)}\tilde{\mathbf{a}}_1(\alpha_q)\tilde{\mathbf{a}}_2^H(\alpha_q),\end{aligned} \quad (12)$$

where $\mathbf{R}_{\mathbf{ss}} = \mathrm{E}[\mathbf{s}(t)\mathbf{s}^H(t)] = \mathrm{diag}([\sigma_1^2,\ldots,\sigma_Q^2])$ is the covariance matrix of the signals where the diagonal entries represent the signal scattering power. In practice, matrix $\mathbf{R}_{\mathbf{x}_{12}}$ is estimated using the T available samples, i.e.,

$$\hat{\mathbf{R}}_{\mathbf{x}_{12}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_1(t)\mathbf{x}_2^H(t). \quad (13)$$

By vectorizing the matrix $\hat{\mathbf{R}}_{\mathbf{x}_{kl}}$, we obtain the following $2MN \times 1$ measurement vector:

$$\mathbf{z} = \mathrm{vec}(\hat{\mathbf{R}}_{\mathbf{x}_{12}}) = \bar{\mathbf{A}}\mathbf{b}, \quad (14)$$

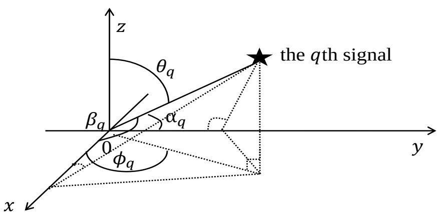

Fig. 2. Illustration on θ_q, ϕ_q, α_q, and β_q.

with

$$\bar{\mathbf{A}} = [\bar{\mathbf{a}}(\alpha_1), \ldots, \bar{\mathbf{a}}(\alpha_Q)], \tag{15}$$

$$\mathbf{b} = [\sigma_1^2 e^{-j\pi\cos(\beta_1)}, \ldots, \sigma_Q^2 e^{-j\pi\cos(\beta_Q)}]^T, \tag{16}$$

where $\bar{\mathbf{a}}(\alpha_q) = \mathbf{a}_1(\alpha_q) \bigotimes \mathbf{a}_2^*(\alpha_q)$ for $1 \leq q \leq Q$. Benefiting from the Vandermonde structure of vectors $\mathbf{a}_1(\alpha_q)$ and $\mathbf{a}_2(\alpha_q)$, the entries in $\bar{\mathbf{a}}(\alpha_q)$ keep the forms of $e^{j\pi(Nm-Mn)\cos(\alpha_q)}$ for $0 \leq m \leq 2M-1$ and $0 \leq n \leq N-1$. Thus, we can regard $\mathbf{z}$ as a virtual received signal from a single snapshot signal vector $\mathbf{b}$ and matrix $\bar{\mathbf{A}}$ that corresponds to a larger virtual array whose virtual sensors are located at the cross-lags between the two subarrays. As such, the number of DOFs is increased, enabling DOA estimation of more signals than the number of sensors.

The signal vector $\mathbf{z}$ in Eqn. (14) can be sparsely represented over the entire discretized angular girds as

$$\mathbf{z} = \mathbf{\Phi}\mathbf{r}, \tag{17}$$

where $\mathbf{\Phi}$ is defined as the collection of steering vectors $\bar{\mathbf{a}}(\alpha_g)$ over all possible grids α_g, $g = 1, \ldots, G$, with $G \gg Q$. It is important to note that $\alpha_q, q = 1, \ldots, Q$, are indicated by positions of the nonzero entries in $\mathbf{r}$, whose values describe the corresponding coefficients.

Compressive sensing (CS) [17] techniques can be used for effective sparse reconstruction and a number of effective algorithms are available [18]–[20]. It is known that CS-based approaches offer a more effective use of the difference coarray sensors obtained from a coprime array in performing DOA estimation as compared to subspace-based methods [15], [21], [22]. As the preferred approach, we use CS algorithms in the Bayesian sparse learning context [23]–[25] as they achieve superior performance and are insensitive to the coherence of dictionary entries. To handle the complex-valued observations and entries in the underlying problem, the complex multitask Bayesian compressive sensing (CMT-BCS) [26] is used to estimate $\hat{\alpha}_q, q = 1, \ldots, Q$, in this paper.

Once α_q, $q = 1, \ldots, Q$, are estimated, the $Q \times 1$ vector $\mathbf{b}$ can be estimated using least squares fitting based on the linear model in Eqn. (14), expressed as

$$\hat{\mathbf{b}} = \left(\hat{\bar{\mathbf{A}}}^H \hat{\bar{\mathbf{A}}}\right)^{-1} \hat{\bar{\mathbf{A}}}^H \mathbf{z}, \tag{18}$$

with $\hat{\bar{\mathbf{A}}} = [\bar{\mathbf{a}}(\hat{\alpha}_1), \ldots, \bar{\mathbf{a}}(\hat{\alpha}_Q)]$. Then, β_q can be obtained as

$$-j\pi\cos(\hat{\beta}_q) = \arg(\hat{b}_q), \tag{19}$$

where $\hat{b}_q$ is the qth element of vector $\hat{\mathbf{b}}$. Note that the estimated parameter $\hat{\beta}_q$ is automatically paired with the corresponding parameter $\hat{\alpha}_q$.

Based on the relationship between (θ_q, ϕ_q) and (α_q, β_q) in Eqns. (5) and (6), the elevation and azimuth angle estimation for each signal can be expressed as

$$\hat{\theta}_q = \sin^{-1}\left[\sqrt{\cos^2(\hat{\alpha}_q) + \cos^2(\hat{\beta}_q)}\right], \tag{20}$$

$$\hat{\phi}_q = \tan^{-1}\left[\frac{\cos(\hat{\alpha}_q)}{\cos(\hat{\beta}_q)}\right]. \tag{21}$$

It is noted that 2-D angle parameters θ_q and ϕ_q are also automatically paired. The proposed approach is simply summarized as follows.

S1: Construct the covariance matrix $\hat{\mathbf{R}}_{\mathbf{x}_{12}}$ with (13).

S2: Obtain the measurements of the virtual array by vectorizing $\hat{\mathbf{R}}_{\mathbf{x}_{12}}$.

S3: Perform CMT-BCS to obtain $\hat{\alpha}_q$ based on (17).

S4: Perform least squares fitting to obtain $\cos(\hat{\beta}_q)$ based on (18) and (19).

S5: Attain the estimates of $\hat{\theta}_q$ and $\hat{\phi}_q$ from (20) and (21).

B. Identifiability

In sparse reconstruction, the Kruskal Rank [28] of $\mathbf{\Phi}$, i.e., Krank($\mathbf{\Phi}$), decides when such recovery is possible. Due to the coprimality between M and N, Krank($\mathbf{\Phi}$)$= 2MN$. Therefore, following the results in [29], the Q-sparse vector $\mathbf{r}$ can be uniquely recovered if and only if

$$Q \leq \frac{\text{Krank}(\mathbf{\Phi})}{2} = MN. \tag{22}$$

Thus, the maximum number of estimated signals $Q_{\max} = MN$. For a given number of physical antennas $2M+N = N_t$, $Q_{\max}$ can be derived by solving the optimization problem:

$$\begin{aligned} \text{Maximize} \quad & Q_{\max} = MN \\ \text{subject to} \quad & 2M + N = N_t, \\ & M < N, \quad M, N \in \mathbb{N}^+. \end{aligned} \tag{23}$$

It is demonstrated in [27] that the valid optimal coprime pair is the one that has $2M$ and N as close as possible. This is satisfied by choosing $N = 2M - 1$. In this case, the maximum number of estimated signals $Q_{\max}$ is

$$Q_{\max} = \frac{N_t^2 - 1}{8}. \tag{24}$$

As shown in Fig. 3, the proposed approach resolves more signals than the number of array sensors when $N_t \geq 8$, whereas the signals to be resolved by the algorithms in [6]–[10] is smaller than the number of antennas.

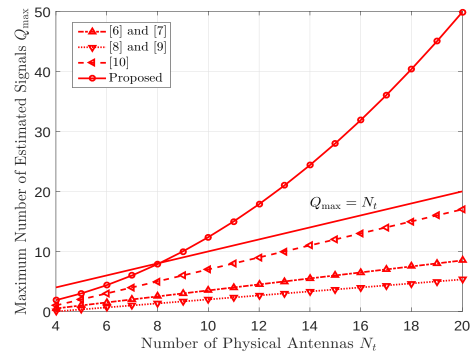

Fig. 3. $Q_{\max}$ versus N_t.

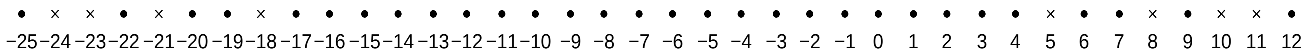

Fig. 4. The resulting coarray ($M = 3$ and $N = 5$; •: virtual sensor positions; ×: holes).

IV. Simulation Results

For illustrative purpose, we consider the 2-D DOA estimation problem using the proposed array configuration consisting of two parallel coprime subarrays, where $M = 3$ and $N = 5$ are assumed. As such, the array configuration consists of $N_t = 2M + N = 11$ antennas. It yields $2MN = 30$ elements in the coarray, as shown in Fig. 4. The increased DOFs can be used to identify up to $(N_t^2 - 1)/8 = MN = 15$ signals, which offer the capability as the 1-D coprime array in [14].

We consider a scenario with $Q = 12$ signals impinging on the array. This number is more than the number of antennas and the available DOFs obtained from the algorithms proposed in [6]–[10]. All signal powers are assumed to be identical and the covariance matrix is obtained by using 2000 snapshots in the presence of noise with a 0 dB input SNR. It is evident that all 12 signals can be identified correctly.

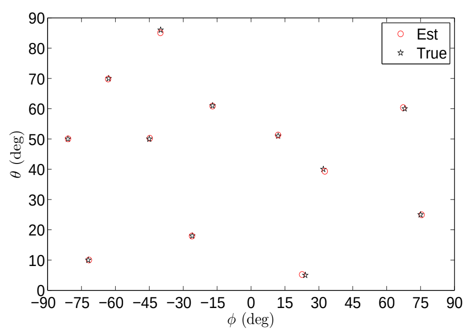

Fig. 5. 2-D DOA estimation results ($Q = 12$).

V. Conclusion

In this paper, a novel algorithm for 2-D DOA estimation was proposed. By exploiting a coprime array configuration with two parallel placed subarrays, the proposed method achieved a larger number of DOFs under the difference coarray equivalence. As such, the proposed technique can resolve more signals than the number of sensors. In the proposed approach, the 2-D DOA estimation was decomposed into two 1-D problems, and the elevation and azimuth angles were paired automatically avoiding any problem with associations. Simulation results showed the effectiveness and the advantages of the proposed method.

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