CHANCE CONSTRAINED BEAMFORMING FOR JOINT RADAR-COMMUNICATION SYSTEMS

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ABSTRACT

We present an intelligent sensor array-based joint radar-communication system which exploits chance constrained programming to develop a robust beamforming design. Probabilistic chance constraints are introduced for the communication operation where the communication objectives are achieved with a desired success rate in the presence of communication channel uncertainties. The chance constraint optimization is then relaxed to form a deterministic and convex problem by employing the statistical profile of the communication channels. Simulation results illustrate the performance of the proposed strategy.

***Index Terms*—** Intelligent sensor arrays, joint radar-communication system, spectrum sharing, robust beamforming, chance constrained programming.

1. INTRODUCTION

Spectrum sharing is enjoying an enormous research attention due to the ever-increasing demand of spectrum resources [1–4]. Significant research efforts have been invested to enable simultaneous operation of multiple applications within the same spectral bands [5–19]. In order to achieve successful operation of co-existing radar and communication systems, it is crucial that both systems cooperate with each other and collaboratively ameliorate their mutual interference. Such challenges can be simplified if both applications are jointly controlled by a single control entity or a physical platform. Joint radar-communication (JRC) systems are examples of such systems where the radar and communication system objectives are achieved by the same physical system [4, 6, 8–17].

There are several popular examples of JRC system configurations in the existing literature. A simple JRC system consists of a single transmitter which exploits dual-purpose radar waveforms [1, 2, 17]. The transfer of communication information is realized by employing different combinations of radar waveforms over the course of a communication interval. More sophisticated JRC systems employ intelligent sensor arrays which enjoy an additional feature of spatial signal multiplexing by exploiting sensor array beamforming to steer dual-purpose waveforms in different directions [4, 6–8, 10, 13–16]. For such systems, the communication operation is enabled by spatially changing the gain and/or phase of the transmitted waveforms towards communication directions or by employing waveform diversity [4, 10, 12–16]. Distributed JRC systems consist of dual-purpose distributed multiple-input multiple-output (MIMO) transducers, which exploit waveform and spatial diversity to carry out both radar and communication operations simultaneously [18].

This paper focuses on JRC systems equipped with an intelligent sensor array which exploits chance constraint-based robust beamforming. In the presence of communication channel uncertainties, such an approach will ensure the communication system quality in a probabilistic sense. Our communication objective will be to achieve the desired communication signal power at communication receivers with a specific probability. For this purpose, we assume non-stationary communication channels such that their statistical profile is known to the intelligent sensor array. Using this statistical profile, we then relax the chance constraints to their equivalent convex deterministic counterparts.

Notations: We use lower-case bold characters to denote vectors. In particular, $|\cdot|$ and $(\cdot)^*$ respectively represent the absolute value and complex conjugate operators. The notation $(.)^{\mathrm{T}}$ and $(.)^{\mathrm{H}}$ respectively denote the transpose and conjugate transpose. In addition, $\mathrm{P}(\cdot)$ denotes the probability operator, whereas $\mathbf{1}_K$ represents the $K \times 1$ column vector of all ones.

2. JOINT RADAR-COMMUNICATION SYSTEM

Consider a JRC system equipped with an M-element linear intelligent sensor array of an arbitrary configuration. There are R single-antenna communication receivers located in the sidelobe region of the radar. The JRC system employs K orthogonal waveforms $\psi_1(t), \psi_2(t), \cdots, \psi_K(t)$ such that

$$\frac{1}{T}\int_0^T \psi_{k_1}(t)\,\psi_{k_2}^*(\zeta)\mathrm{d}t = \delta(k_1 - k_2)\,\delta(t-\zeta), \qquad (1)$$

where $1 \leq k_1, k_2 \leq K$, t is the fast time, T is the pulse duration, $\psi_{k_2}(\zeta)$ is the time delayed version of $\psi_{k_1}(t)$ delayed by ζ ($< T$), and $\delta(\cdot)$ represents the Kronecker delta function.

Both radar and communication operations are performed by the same transmit array exploiting their respective waveforms $\psi_{\mathrm{rad}}(t)$ and $\psi_{\mathrm{com}}(t)$. The mutual interference between the radar and communication systems is mitigated by employing orthogonal waveforms and spatial filtering. Similar

This work is supported in part by the National Science Foundation (NSF) under grant AST-1547420.

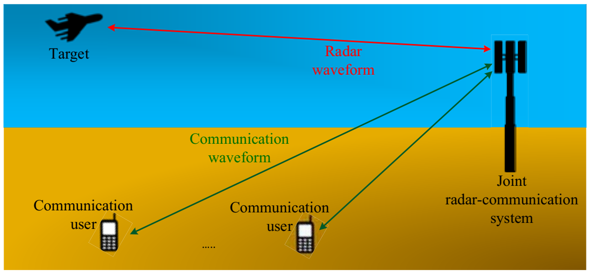

Fig. 1. Joint radar-communication system

to [20], we assume that $\psi_{\rm rad}(t) = \psi_1(t)$ and $\psi_{\rm com}(t)$ is selected from the remainder of the $K-1$ orthogonal waveforms depending on which information is transmitted.

2.1. Beamformer Design

During each radar pulse, the JRC system exploits two beamforming weight vectors $\mathbf{w}_{\rm rad}$ and $\mathbf{w}_{\rm com}$ which correspond to the waveforms $\psi_{\rm rad}(t)$ and $\psi_{\rm com}(t)$, respectively.

Denote $\mathbf{a}(\theta)$ as the array response vector of the transmit JRC array in the direction θ, and $\theta_{\rm rad}$ as the direction of the radar main lobe, whereas $\Theta_{\rm com}$ is the set containing the directions of all the communication users. Note that all the communication users are located in the sidelobe of the radar, i.e., $\Theta_{\rm com} \subset \Theta^{\rm c}_{\rm rad}$, where $\Theta^{\rm c}_{\rm rad}$ represents the radar sidelobe region which excludes the main lobe and its corresponding transition region.

The beamforming weight vectors $\mathbf{w}_{\rm rad}$ and $\mathbf{w}_{\rm com}$ can be designed as:

$$\begin{aligned}
\min_{\mathbf{w}_{\rm rad},\mathbf{w}_{\rm com}} \quad & \mathbf{w}^{\rm H}_{\rm rad}\mathbf{w}_{\rm rad} + \mathbf{w}^{\rm H}_{\rm com}\mathbf{w}_{\rm com} \\
\text{subject to} \quad & \mathbf{w}^{\rm H}_{\rm rad}\mathbf{a}(\theta_{\rm rad}) = 1, \\
& \mathbf{w}^{\rm H}_{\rm rad}\mathbf{a}(\theta_c) = 0, \qquad \theta_c \in \Theta_{\rm com}, \\
& \mathbf{w}^{\rm H}_{\rm rad}\mathbf{a}(\theta_{\rm rad,sl}) \le \alpha_{\rm rad}, \qquad \theta_{\rm rad,sl} \in \Theta^{\rm c}_{\rm rad}, \\
& \mathbf{w}^{\rm H}_{\rm com}\mathbf{a}(\theta_{\rm rad}) = 0, \\
& \mathbf{w}^{\rm H}_{\rm com}\mathbf{a}(\theta_c) \ge \Delta_c, \qquad \theta_c \in \Theta_{\rm com},
\end{aligned} \tag{2}$$

where $\alpha_{\rm rad}$ denotes the worst-case amplitude level of the radar waveform towards all the angles $\theta_{\rm rad,sl}$ in the radar sidelobe region $\Theta^{\rm c}_{\rm rad}$ and $\Delta_c > 0$ is the desired communication amplitude transmitted towards the cth communication receiver. Due to the power minimization objective of the above optimization, the communication amplitudes will always approach Δ_c, i.e., $\mathbf{w}^{\rm H}_{\rm com}\mathbf{a}(\theta_c) = \Delta_c$ (note that the imaginary part is equal to zero). Since $\mathbf{w}_{\rm rad}$ and $\mathbf{w}_{\rm com}$ are designed to be orthogonal in the radar and communications directions, we may choose Δ_c to be higher than $\alpha_{\rm rad}$ without compromising the radar operation. Note that, as Δ_c is real, the imaginary part of $\mathbf{w}^{\rm H}_{\rm com}\mathbf{a}(\theta_c)$ approaches zero.

2.2. Signalling Strategy

The composite signal transmitted from the JRC platform during each radar pulse is represented as [20]:

$$\mathbf{x}(t,\tau) = \mathbf{w}^*_{\rm rad}\psi_{\rm rad}(t) + \mathbf{w}^*_{\rm com}\psi_{\rm com}(t), \tag{3}$$

where τ is the slow-time index, $\psi_{\rm rad}(t) = \psi_1(t)$, and $\psi_{\rm com}(t)$ is given by

$$\psi_{\rm com}(t) = \boldsymbol{\beta}^{\rm T}(\tau)\mathbf{\Psi}(t). \tag{4}$$

Here, $\mathbf{\Psi}(t) = [\psi_2(t), \psi_3(t), \cdots, \psi_K(t)]^{\rm T}$ is the dictionary of communication waveforms and is assumed to be known at each communication receiver. $\boldsymbol{\beta}(\tau)$ is a $(K-1)\times 1$ binary selection vector which specifies the desired communication waveform from the dictionary $\mathbf{\Psi}(t)$ for each slow-time index given that $\boldsymbol{\beta}^{\rm T}(\tau)\mathbf{1}_{K-1} = 1, \forall\tau$.

For the time-invariant communication channels, i.e., the channels do not change with the slow time τ, we denote the channel gain between the JRC transmitter and the cth communication receiver as h_c. Then, the received signal at the cth communication user takes the following form:

$$\begin{aligned}
s_c(t,\tau) &= h_c\mathbf{x}^{\rm T}(t)\mathbf{a}(\theta_c) + n_c(t) \\
&= h_c\tilde{\Delta}_c\psi_{\rm com}(t) + n_c(t),
\end{aligned} \tag{5}$$

where $\tilde{\Delta}_c \ge \Delta_c$ and $n_c(t)$ is the additive complex white Gaussian noise

The communication information is extracted at the communication receivers by estimating the modulated waveform transmitted by the JRC transmit array during each radar pulse. This is performed by matched filtering of the received signal in Eq. (5) with all the communication waveforms as:

$$\begin{aligned}
r_c(\tau) &= \frac{1}{T}\int_{t=0}^{T} s\left(t,\tau\right)\psi_k\left(t\right)dt \\
&= \begin{cases} h_c\tilde{\Delta}_c + \tilde{n}_c(\tau), & \text{if } \psi_k(t) \text{ was transmitted}, \\ \tilde{n}_c(\tau), & \text{otherwise}, \end{cases}
\end{aligned} \tag{6}$$

where $r_c(\tau)$ is the output of the matched filter during the slow-time index τ and $\tilde{n}_c(\tau)$ is the corresponding noise output at the cth communication receiver.

3. CHANCE CONSTRAINED BEAMFORMING DESIGN FOR JRC SYSTEM

In this section, we present chance constrained beamforming for the JRC system under Rayleigh fading communication channels. Our objective is to optimize the communication performance of the system by incorporating robustness in the beamformer design against communication channel uncertainties through the exploitation of chance constraints. The resulting nonlinear optimization is further relaxed into a convex form by employing the information of probability density function (PDF) of the channel conditions.

3.1. Incorporating Robustness through Chance Constraint

We assume that the magnitude of the communication channel gain for different radar pulses follows the Rayleigh distribution, i.e. the communication channels vary with the slow time τ, such that $|h_c(\tau)| = \bar{h}_c\tilde{h}_c, \forall\tau$, where $\bar{h}_c$ is a constant accounting for the propagation loss and $\tilde{h}_c \sim \mathcal{R}(\sigma_c)$ with $\mathcal{R}(\sigma_c)$

denoting Rayleigh distribution with scale parameter (mode) of σ_c. Such a model is relevant as long as the large-scale channel parameters remain constant. Note that $|h_c(\tau)|$ will follow $\mathcal{R}(\bar{h}\sigma_c)$.

At the cth communication receiver, the required minimum signal amplitude is $\bar{\Delta}_c = \bar{h}_c\Delta_c$. This amplitude requirement is satisfied by (2) for non-fading channels. For fading channels, however, the communication channel gain $|h_c(\tau)|$ is a stochastic process, and the worse-case value of the received signal amplitude at the cth communication receiver, $\Delta_c|h_c(\tau)| = \bar{\Delta}_c|h_c(\tau)|/\bar{h}_c$, varies over time. Therefore, the desired communication objective is ensured only if $|h_c(\tau)|/\bar{h}_c = \tilde{h}_c \geq 1$ holds. As $|h_c(\tau)| \sim \mathcal{R}(\bar{h}_c\sigma_c)$, the achieved signal amplitude at the cth communication receiver can fall below the desired amplitude with a probability $\mathrm{P}(|h_c(\tau)| < \bar{h}_c) = \mathrm{P}(\tilde{h}_c < 1)$. This illustrates the sub-optimal performance exhibited by the optimization (2) and emphasizes a need for robust design which incorporates these channel uncertainties.

In order to maintain the communication signal level to be higher than the desired amplitude with a required probability, we employ chance constrained optimization as follows:

$$\begin{aligned} \min_{\mathbf{w}_{\mathrm{rad}},\mathbf{w}_{\mathrm{com}}} \quad & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{w}_{\mathrm{rad}} + \mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{w}_{\mathrm{com}}, \\ \text{subject to} \quad & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{a}(\theta_{\mathrm{rad}}) = 1, \\ & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{a}(\theta_c) = 0, \qquad \theta_c \in \Theta_{\mathrm{com}}, \\ & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{a}(\theta_{\mathrm{rad,sl}}) \leq \alpha_{\mathrm{rad}}, \qquad \theta_{\mathrm{rad,sl}} \in \Theta_{\mathrm{rad}}^{\mathrm{c}}, \\ & \mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_{\mathrm{rad}}) = 0, \\ & \mathrm{P}\left(\mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_c)\tilde{h}_c \geq \Delta_c\right) \geq \eta, \quad \theta_c \in \Theta_{\mathrm{com}}, \end{aligned} \tag{7}$$

where η is the desired probability ensuring the quality of service such that the constraint $\mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_c)\tilde{h}_c \geq \Delta_c$ should be true. Since $\mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_c)$ has a zero imaginary part, $\mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_c)\tilde{h}_c$ always represents the real transformation of the Rayleigh random variable $\tilde{h}_c$.

The optimization problem (7) ensures that we achieve the received communication signal level higher than the desired amplitude with a probability η. Such strategy is practical as it will subsequently result in a controlled bit error rate (BER) for the communication system by ensuring the desired signal power at the communication receivers.

Note that, if we directly modify the optimization problem (2) by replacing the last constraint by $\mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_c) \geq \Delta_c/\tilde{h}_c$, such strategy will try to ensure the desired signal level of $\bar{\Delta}_c$ even if the communication channel undergoes deep fading, resulting in significant power loss. In contrary, the proposed strategy (7) ameliorates this requirement by ensuring the communication performance for the $\eta \times 100\%$ of the communication interval. This implies that the chance constraints will not be satisfied for $(1-\eta) \times 100\%$ of the slow time indexes in the worst channel conditions (left tail of Rayleigh distribution where channel gain is significantly low), thus impeding unnecessary power loss. Practically, the small probability of unsatisfactory signal amplitude is compensated by channel coding to render the desirable BER performance [21].

3.2. Convex Relaxation

The chance constraint-based optimization in (7) is difficult to solve due to its nonlinearity and the dynamic behavior of the communication channel gain. In the following, we relax this chance constraint into a deterministic constraint by employing the statistical information of the communication channel gain. For this purpose, it is assumed that the PDFs of the communication channels are either known or can be obtained for the chance constraint problem under consideration.

Theorem 1: Denote $\Phi(u) = 1 - \mathrm{e}^{-u^2/2}$ as the cumulative distribution function (CDF) of $u \sim \mathcal{R}(1)$, and $\Phi^{-1}(\eta) = [-2\ln(1-\eta)]^{0.5}$ as the inverse function of $\Phi(u)$, where η is the probability. Then, for a Rayleigh random variable $a \sim \mathcal{R}(\sigma_a)$, the chance constraint $\mathrm{P}\{ya \geq b\} \geq \eta$ is equivalent to $y\sigma_a\Phi^{-1}(1-\eta) \geq b$ where y and b are positive constants.

Proof: Let $\Phi_a(a)$ denote the CDF of a. We can write

$$\mathrm{P}\{ya \geq b\} = \mathrm{P}\{a \geq b/y\} = 1 - \Phi_a(b/y). \tag{8}$$

Because a follows the distribution $\mathcal{R}(\sigma_a)$, its CDF is given by $\Phi_a(a) = 1 - \mathrm{e}^{-a^2/(2\sigma_a^2)}$. The corresponding inverse function of $\Phi_a(a)$ takes the form $\Phi_a^{-1}(\eta) = \sigma_a\Phi^{-1}(\eta)$.

The chance constraint under consideration subsequently takes the following forms:

$$\begin{aligned} & \mathrm{P}\{ya \geq b\} \geq \eta \\ \Longrightarrow \quad & 1 - \Phi_a(b/y) \geq \eta \\ \Longrightarrow \quad & y\Phi_a^{-1}(1-\eta) \geq b \\ \Longrightarrow \quad & y\sigma_a\Phi^{-1}(1-\eta) \geq b. \end{aligned}$$

Note that $\Phi^{-1}(1-\eta)$ is always positive because the desired probability always follows $0 \leq \eta \leq 1$ for $0 \leq u \leq \infty$. Moreover, $\sigma_a\Phi^{-1}(1-\eta)$ is a constant which makes the above constraint deterministic and linear. □

In the above theorem, we see that the chance constraint can be relaxed into a deterministic constraint using the PDF for the JRC case under consideration. Using this theorem, we replace the chance constraint in (7) by the deterministic convex (linear) constraint which results in the following convex optimization formulation:

$$\begin{aligned} \min_{\mathbf{w}_{\mathrm{rad}},\mathbf{w}_{\mathrm{com}}} \quad & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{w}_{\mathrm{rad}} + \mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{w}_{\mathrm{com}}, \\ \text{subject to} \quad & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{a}(\theta_{\mathrm{rad}}) = 1, \\ & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{a}(\theta_c) = 0, \\ & \mathbf{w}_{\mathrm{rad}}^{\mathrm{H}}\mathbf{a}(\theta_{\mathrm{rad,sl}}) \leq \alpha_{\mathrm{rad}}, \\ & \mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_{\mathrm{rad}}) = 0, \\ & \mathbf{w}_{\mathrm{com}}^{\mathrm{H}}\mathbf{a}(\theta_c)\sigma_c\Phi^{-1}(1-\eta) \geq \Delta_c. \end{aligned} \tag{9}$$

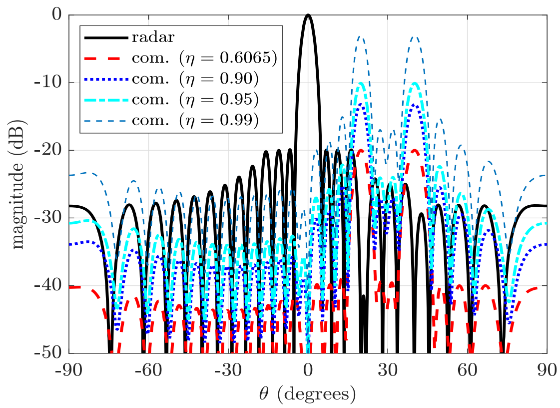

Fig. 2. Beamforming patterns for different desired probabilities η ($M = 25, \theta_{\text{rad}} = 0^{\text{o}}, \theta_1 = 20^{\text{o}}, \theta_2 = 40^{\text{o}}, \Delta_c = 0.1, \alpha_{\text{rad}} = 0.1, \sigma_c = 1$).

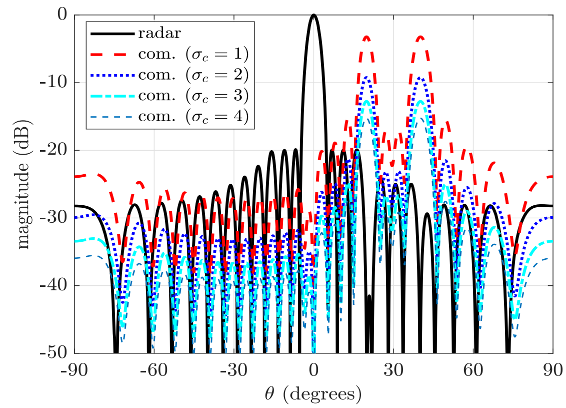

Fig. 3. Beamforming patterns for different channel scale parameters σ_c ($M = 25, \theta_{\text{rad}} = 0^{\text{o}}, \theta_1 = 20^{\text{o}}, \theta_2 = 40^{\text{o}}, \Delta_c = 0.1, \alpha_{\text{rad}} = 0.1, \eta = 0.9$).

In practice, we are usually interested in $\eta \geq 0.9$ for efficient communication. Several different values of η are considered in the simulation evaluations.

It is interesting to observe that for $\sigma_c = 1$, if $\eta = 0.6065$, $\Phi^{-1}(1-\eta) = 1$ and the optimization (9) becomes exactly the same as optimization (2). This implies that the solution to the optimization (2) ensures the efficient communication only for 60.65% of the communication time.

4. SIMULATION RESULTS

In this section, we present simulation results illustrating the performance of the JRC system exploiting the proposed chance constrained optimization. In all simulation examples, the transmit JRC system is equipped with a 25-element uniform linear array and the interelement spacing is half a wavelength. The radar main beam is directed towards $\theta_{\text{rad}} = 0^{\text{o}}$, whereas two communication users are located at $\theta_1 = 20^{\text{o}}$ and $\theta_2 = 40^{\text{o}}$, respectively. The desired amplitude of the communication signal towards both communication users is assumed to be $\Delta_c = 0.1$. The maximum allowable sidelobe level for radar waveform is $\alpha_{\text{rad}} = 0.1$. We use the SDPT3 solver [22] with the CVX [23] toolbox for solving all the optimization problems.

In the first simulation, we assume $\sigma_c = 1$ for all the communication channels. The communication beampatterns have been plotted for the cases of $\eta = 0.6065, 0.9, 0.99$ and 0.999. Fig. 2 illustrates the beampatterns extracted for the radar and communication signals by employing the optimization (9). Note that the radar beampattern is the same for all cases and its amplitude is below the desired sidelobe levels in its sidelobe regions. Because the radar and communication waveforms are orthogonal to each other, their mutual interference between the radar and communication directions is small, i.e., the radar beampattern has nulls towards the communication directions, and vice versa.

Now we consider the impact of the different values of probability η to the amplitudes of communication beamformers in the directions of communication receivers. It is observed that, in order to achieve the communication objective with a higher probability, higher communication power is transmitted in the direction of communication users. The results shown for $\eta = 0.6065$ render the results of optimization problem (2). In this case, the power utilization is low, and the communication objective is achieved only for 60.65% of the slow time indexes, corresponding to worse communication performance among the results being compared here.

In the second simulation, we fix the probability η to be 0.9. The scale parameter of Rayleigh distribution σ_c for underlying communication channels varies in this simulation for different beamforming weight vectors. However, both communication users experience the same channel conditions. It is observed again in Fig. 3 that radar and communication beampatterns minimize their mutual interference. Moreover, as the scale parameter of communication channels increases, less communication power is required to ensure the success probability of $\eta = 0.9$ for the communication objectives. This is because an increase in the scale parameter for Rayleigh distribution results in an increase in the mean of the distribution which corresponds to higher channel gains.

5. CONCLUSION

In this paper, we present the chance constrained programming-based optimization strategy for JRC system. We introduce probabilistic constraints for the communication operation which optimize the transmit power according to the channel conditions and prevent the drain of communication power in case of momentous deep fades. It is also observed that we need more communication power for the cases where communication channels have lower gain or if a high communication success rate is required. Simulation results illustrate the performance of the proposed strategy.

6. REFERENCES

[1] C. Sturm, T. Zwick, and W. Wiesbeck, "An OFDM system concept for joint radar and communications operations," in *Proc. IEEE Veh. Technol. Conf.*, Barcelona, Spain, Apr. 2009.

[2] S. D. Blunt, M. R. Cook, and J. Stiles, "Embedding information into radar emissions via waveform implementation," in *Proc. Waveform Diversity and Design Conf.*, Niagara Falls, Canada, Aug. 2010, pp. 195–199.

[3] H. Griffiths, S. Blunt, L. Cohen, and L. Savy, "Challenge problems in spectrum engineering and waveform diversity," in *Proc. IEEE Radar Conf.*, Ottawa, Canada, Apr.–May 2013, pp. 1–5.

[4] A. Hassanien, M. G. Amin, Y. D. Zhang, and F. Ahmad, "Signaling strategies for dual-function radar communications: An overview," *IEEE Aerosp. Electron. Syst. Mag.*, vol. 31, no. 10, pp. 36–45, Oct. 2016.

[5] D. W. Bliss, "Cooperative radar and communications signaling: The estimation and information theory odd couple," in *Proc. IEEE Radar Conf.*, Cincinnati, OH, May 2014, pp. 50–55.

[6] J. Euzière, R. Guinvarc'h, M. Lesturgie, B. Uguen, and R. Gillard, "Dual function radar communication time-modulated array," in *Proc. Int. Radar Conf.*, Lille, France, Oct. 2014, pp. 1–4.

[7] Z. Geng, H. Deng, and B. Himed, "Adaptive radar beamforming for interference mitigation in radar-wireless spectrum sharing," *IEEE Signal Process. Lett.*, vol. 22, no. 4, pp. 484–488, Apr. 2015.

[8] J. R. Guerci, R. M. Guerci, A. Lackpour, and D. Moskowitz, "Joint design and operation of shared spectrum access for radar and communications," in *Proc. IEEE Radar Conf.*, Arlington, VA, May 2015, pp. 761–766.

[9] A. Hassanien, M. G. Amin, Y. D. Zhang, and F. Ahmad, "Dual-function radar-communications using phase-rotational invariance," in *Proc. European Signal Process. Conf.*, Nice, France, Aug. 2015, pp. 1346–1350.

[10] A. Hassanien, M. G. Amin, Y. D. Zhang, and F. Ahmad, "Dual-function radar-communications: Information embedding using sidelobe control and waveform diversity," *IEEE Trans. Signal Process.*, vol. 64, no. 8, pp. 2168–2181, Apr. 2016.

[11] A. Hassanien, M. G. Amin, Y. D. Zhang, and F. Ahmad, "Phase-modulation based dual-function radar-communications," *IET Radar Sonar & Nav.*, vol. 10, no. 10, pp. 1411–1421, 2016.

[12] Y. Liu, G. Liao, J. Xu, Z. Yang, and Y. Zhang, "Adaptive OFDM integrated radar and communications waveform design based on information theory," *IEEE Commun. Lett.*, vol. 21, no. 10, pp. 2174–2177, Oct. 2017.

[13] A. Ahmed, Y. D. Zhang, and B. Himed, "Multi-user dual-function radar-communications exploiting sidelobe control and waveform direversity," in *Proc. IEEE Radar Conf.*, Oklahoma City, OK, Apr. 2018.

[14] F. Liu, L. Zhou, C. Masouros, A. Li, W. Luo, and A. Petropulu, "Toward dual-functional radar-communication systems: Optimal waveform design," *IEEE Trans. Signal Process.*, vol. 66, no. 16, pp. 4264–4279, Aug. 2018.

[15] A. Ahmed, Y. Gu, D. Silage, and Y. D. Zhang, "Power-efficient multi-user dual-function radar-communication," in *Proc. IEEE Int. Workshop Signal Process. Advances in Wireless Commun.*, Kalamata, Greece, June 2018, pp. 1–5.

[16] A. Ahmed, Y. D. Zhang, and Y. Gu, "Dual-function radar-communications using QAM-based sidelobe modulation," *Digital Signal Process.*, vol. 82, pp. 166–174, Nov. 2018.

[17] M. Bică and V. Koivunen, "Multicarrier radar-communications waveform design for RF convergence and coexistence," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Brighton, U.K., May 2019, pp. 7780–7784.

[18] A. Ahmed, Y. D. Zhang, and B. Himed, "Distributed dual-function radar-communication MIMO system with optimized resource allocation," in *Proc. IEEE Radar Conf.*, Boston, MA, Apr. 2019.

[19] A. Ahmed, Y. D. Zhang, A. Hassanien, and B. Himed, "OFDM-based joint radar-communication system: Optimal sub-carrier allocation and power distribution by exploiting mutual information," in *Proc. Asilomar Conf. Signals, Syst., Comput.*, Pacific Grove, CA, Nov. 2019.

[20] X. Wang, A. Hassanien, and M. G. Amin, "Sparse transmit array design for dual-function radar communications by antenna selection," *Digital Signal Process.*, vol. 83, pp. 223–234, Dec. 2018.

[21] A. Goldsmith, *Wireless Communications*, Cambridge Univ. Press, 2005.

[22] R. H. Tütüncü, K. C. Toh, and M. J. Todd, "Solving semidefinite-quadratic-linear programs using SDPT3," *Math. Program.*, vol. 95, no. 2, pp. 189–217, Feb. 2003.

[23] CVX Research, "CVX: Matlab software for disciplined convex programming, version 2.0," `http://cvxr.com/cvx`, Aug. 2012.