COPRIME ARRAY DESIGN WITH MINIMUM LAG REDUNDANCY

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ABSTRACT

In this paper, we propose a new sparse coprime array design that achieves a higher number of degrees-of-freedom for direction-of-arrival (DOA) estimation. The proposed array design completely avoids lag redundancies between the two constituting subarrays of the coprime array, thus achieving the maximum number of unique correlation lags under the coprime array framework. As a result, given the same number of physical sensors, the proposed design resolves more sources than other coprime array designs with enhanced DOA estimation performance. Simulation results demonstrate the superior performance of the proposed coprime array design.

***Index Terms*—** Sparse arrays, coprime array, difference co-array, direction-of-arrival estimation, compressive sensing.

1. INTRODUCTION

Sparse arrays are attractive in direction-of-arrival (DOA) estimation due to their capability to achieve enhanced performance without increasing the number of physical sensors [1, 2]. More specifically, by using a sparse array, in lieu of a uniform linear one, a higher number of degrees of freedom (DOFs) can be obtained in the context of difference co-arrays. Among various sparse array design schemes, coprime array is one of the desirable choices that enable systematical sparse array design and has attracted significant interest in the recent years [3–13].

While the earlier approaches mainly rely on the subspace-based DOA estimation methods (such as MUSIC and ESPRIT) which require spatial smoothing to restore the rank of the signal covariance matrix in the co-array context and can only utilize consecutive lags [4, 7], many recent works are based on sparse reconstruction so as to use all the unique lags, whether they are consecutive or not [6, 14, 15]. Other approaches to achieve a higher number of DOFs include the use of wideband signals [16,17], signals with two or more frequencies [18–20], higher-order statistics [21–24], and virtual array interpolation [25–27].

For a given number of array sensors in a coprime array, the achievable number of unique lags is determined by the level of redundancies in the yielded difference co-array. In other words, the number of DOFs can be improved by reducing the number of lag redundancies. Because a coprime array uses two constituting uniform linear subarrays, the redundancies in the auto-lags within each subarray cannot be avoided. Therefore, noticing that the auto-lags is a subset of the cross-lags between the two subarrays, the objective becomes the reduction of the cross-lags [6]. The maximum number of unique lags is thus achieved in the context of coprime array when cross-lag redundancies are completely eliminated.

In this paper, we examine the conditions for cross-lags to coincide and, based on this observation, propose sparse array designs within the coprime array framework that completely eliminate cross-lag redundancies between constituting subarrays. As such, the proposed array designs achieve the highest possible number of unique lags for a given number of physical sensors in the context of coprime arrays. Simulation examples are presented to demonstrate the effectiveness of the proposed sparse array designs.

Notations: We use lower-case (upper-case) bold characters to denote vectors (matrices). In particular, $\mathbf{I}_N$ and $\mathbf{0}_N$ denote the $N \times N$ identity and null matrices, respectively. $(.)^{\mathrm{T}}$ and $(.)^{\mathrm{H}}$ respectively denote the transpose and conjugate transpose of a matrix or vector, $\mathrm{vec}(\cdot)$ is the vectorization operator that turns a matrix into a vector by stacking all columns on top of the another, and $\mathrm{diag}(\mathbf{x})$ denotes a diagonal matrix that uses the elements of $\mathbf{x}$ as its diagonal elements. In addition, $\mathrm{E}[\cdot]$ represents the statistical expectation operator, whereas $\otimes$ denotes the Kronecker product.

2. SIGNAL MODEL

Consider a sparse sensor array consisting of Q antennas located at $p_0 \cdot \lambda/2, \ \ldots, \ p_{Q-1} \cdot \lambda/2$, where λ is the signal wavelength. The values of $p_0, \ \cdots, \ p_{Q-1}$ are assumed to be integers. Without loss of generality, the first sensor position is considered as the reference, i.e., $p_0 = 0$.

When K uncorrelated narrow-band far-field signals impinge on the array from angles $\theta_1, \ldots, \theta_K$, the data vector received at the antenna array is expressed as:

$$\mathbf{x}(t) = \sum_{k=1}^{K} s_k(t)\mathbf{a}(\theta_k) + \mathbf{n}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \qquad (1)$$

where $s_k(t)$ denotes the baseband waveform of the kth signal, and $\mathbf{a}(\theta_k) = [1, e^{j\pi p_1 \sin(\theta_k)}, \ldots, e^{j\pi p_{Q-1} \sin(\theta_k)}]^{\mathrm{T}}$ is the array steering vector corresponding to angle θ_k. The array manifold matrix is given as $\mathbf{A} = [\mathbf{a}(\theta_1), \ \ldots, \mathbf{a}(\theta_K)]$, and $\mathbf{s}(t) = [s_1(t), \ldots, s_K(t)]^{\mathrm{T}}$ denotes the signal vector. The elements of the noise vector $\mathbf{n}(t) \sim \mathcal{CN}(\mathbf{0}_Q, \sigma_n^2 \mathbf{I}_Q)$ are assumed

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to be independent and identically distributed (i.i.d.) complex white Gaussian random processes.

The covariance matrix of data vector $\mathbf{x}(t)$ is obtained as:

$$\mathbf{R_x} \;=\; \mathrm{E}[\mathbf{x}(t)\mathbf{x}^{\mathrm{H}}(t)] = \mathbf{A}\mathbf{R_s}\mathbf{A}^{\mathrm{H}} + \sigma_n^2\mathbf{I}_Q, \tag{2}$$

where $\mathbf{R}_s = \mathrm{E}[\mathbf{s}(t)\mathbf{s}^H(t)] = \mathrm{diag}([\sigma_1^2, \dots, \sigma_K^2])$ is the source covariance matrix with σ_k^2 denoting the power of the kth source.

2.1. Difference Co-arrays

Vectorizing $\mathbf{R_x}$ yields

$$\mathbf{z} = \mathrm{vec}(\mathbf{R_x}) = \tilde{\mathbf{A}}\mathbf{b} + \sigma_n^2\tilde{\mathbf{i}}, \tag{3}$$

where $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}(\theta_1), ..., \tilde{\mathbf{a}}(\theta_K)]$, $\tilde{\mathbf{a}}(\theta_k) = \mathbf{a}^*(\theta_k) \otimes \mathbf{a}(\theta_k)$, $\mathbf{b} = [\sigma_1, ..., \sigma_K]^{\mathrm{T}}$, and $\tilde{\mathbf{i}} = \mathrm{vec}(\mathbf{I}_Q)$. Note that vector $\mathbf{z}$ amounts to the received data from a virtual array with a much larger aperture defined by the virtual steering matrix $\tilde{\mathbf{A}}$ having the co-array lag locations. The DOFs of any sparse array configuration is determined by $(\eta+1)/2$ where η is the achieved number of correlation lags.

The entries of the covariance matrix $\mathbf{R_x}$ correspond to difference correlation lags. Denote $\mathcal{P} = \{p_0, \cdots, p_{Q-1}\}$ as an integer set representing the sensor positions, the difference co-array $\mathcal{D}$ containing all the possible lags are expressed as:

$$\mathcal{D} = \mathcal{P} \ominus \mathcal{P} = \bigcup_{\forall p_l \in \mathcal{P}, \forall p_k \in \mathcal{P}} \{p_l - p_k\}. \tag{4}$$

2.2. Prototype Coprime Array

Consider a prototype coprime array [5], as illustrated in Fig. 1, which consists of a pair of uniform linear subarrays. One of the subarrays uses M sensors with an inter-element spacing of $N \cdot \lambda/2$, whereas the other subarray uses N elements with an inter-element spacing of $M \cdot \lambda/2$. M and N are chosen to be coprime integers. The sensor position sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of the two subarrays, described in terms of the integer multiples of half-wavelength, are respectively expressed as:

$$\begin{aligned} \mathcal{S}_1 &= \{nM \mid 0 \le n \le N-1\}, \\ \mathcal{S}_2 &= \{mN \mid 0 \le m \le M-1\}. \end{aligned} \tag{5}$$

Since the two subarrays share the first sensor at the zeroth position, the resulting $Q = M + N - 1$ elements of the coprime array are positioned at:

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_1 \cup \mathcal{S}_2 \\ &= \{nM \mid 0 \le n \le N-1\} \cup \{mN \mid 0 \le m \le M-1\}. \end{aligned} \tag{6}$$

The difference co-array $\mathcal{D}$ of the coprime array having the sensor location set $\mathcal{S}$ is given as:

$$\mathcal{D} = \mathcal{S} \ominus \mathcal{S} = (\mathcal{S}_1 \ominus \mathcal{S}_1) \cup (\mathcal{S}_1 \ominus \mathcal{S}_2) \cup (\mathcal{S}_2 \ominus \mathcal{S}_1) \cup (\mathcal{S}_2 \ominus \mathcal{S}_2). \tag{7}$$

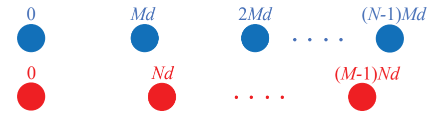


Fig. 1. Coprime array configuration.

Here, $(\mathcal{S}_1 \ominus \mathcal{S}_1)$ and $(\mathcal{S}_2 \ominus \mathcal{S}_2)$ are referred to as the auto-lag sets for the two subarrays, whereas $(\mathcal{S}_1 \ominus \mathcal{S}_2)$ and $(\mathcal{S}_2 \ominus \mathcal{S}_1)$ form the cross-lag sets between the two subarrays. These sets are respectively given as

$$\begin{aligned} \mathcal{S}_1 \ominus \mathcal{S}_1 &= \{(n_1 - n_2)M\} = \{l_1 M\}, \\ \mathcal{S}_1 \ominus \mathcal{S}_2 &= \{(nM - mN)\}, \\ \mathcal{S}_2 \ominus \mathcal{S}_1 &= \{(mN - nM)\}, \\ \mathcal{S}_2 \ominus \mathcal{S}_2 &= \{(m_1 - m_2)N\} = \{l_2 N\}, \end{aligned} \tag{8}$$

where $0 \le \{n, n_1, n_2\} \le N-1$, $0 \le \{m, m_1, m_2\} \le M-1$, $-N+1 \le l_1 \le N-1$, and $-M+1 \le l_2 \le M-1$.

Existing coprime array structures contain several self-lag as well as cross-lag redundancies. The self-lag redundancies in sets $\mathcal{S}_1 \ominus \mathcal{S}_1$ and $\mathcal{S}_2 \ominus \mathcal{S}_2$ are evident and cannot be avoided because the two subarrays are uniform linear. On the other hand, cross-lag redundancies arise only when overlapping elements exist in sets $\mathcal{S}_1 \ominus \mathcal{S}_2$ and $\mathcal{S}_2 \ominus \mathcal{S}_1$. Note that these two cross-lag sets are symmetrical.

3. PROPOSED ARRAY DESIGN WITH NO CROSS-LAG REDUNDANCY

In this section, we propose a modified coprime array design to eliminate cross-lag redundancies. We first examine the conditions for cross-lag redundancies to exist in the prototype coprime array design, and then propose sparse array designs with no cross-lag redundancies.

3.1. Cross-lag redundancy conditions

Proposition 1: Each of the cross-lag sets $\mathcal{S}_1 \ominus \mathcal{S}_2$ and $\mathcal{S}_2 \ominus \mathcal{S}_1$ contains MN unique elements.

Proof. Because $\mathcal{S}_1 \ominus \mathcal{S}_2$ and $\mathcal{S}_2 \ominus \mathcal{S}_1$ are symmetric, we only provide the proof for the former. Is it observed from Eq. (8) set $\mathcal{S}_1 \ominus \mathcal{S}_2$ has MN unique elements unless the following relation holds for two *different* pairs of $(m_1, n_1) \neq (m_2, n_2)$:

$$n_1 M - m_1 N = n_2 M - m_2 N, \tag{9}$$

or, equivalently,

$$(n_1 - n_2)M = (m_1 - m_2)N, \tag{10}$$

where $0 \le \{n_1, n_2\} \le N-1$, $0 \le \{m_1, m_2\} \le M-1$. Since M and N are coprime, the above condition is satisfied only if: (a) $n_1 - n_2 = 0$ and $m_1 - m_2 = 0$. This condition contradicts with the different pair requirement; and (b) $n_1 - n_2 = kN$ and $m_1 - m_2 = kM$ where k is a positive integer. This condition cannot be satisfied because $\{m_1, m_2\} \le M-1$ and $\{n_1, n_2\} \le N-1$. Therefore, $\mathcal{S}_1 \ominus \mathcal{S}_2$ (and, similarly, $\mathcal{S}_2 \ominus \mathcal{S}_1$) contains MN unique elements. $\square$

Proposition 2: There are $MN - M - N + 2$ cross-lag redundancies in the prototype coprime array.

Proof. From Eq. (8), it is observed that elements in set $\mathcal{S}_1 \ominus \mathcal{S}_2$ overlap with those in $\mathcal{S}_2 \ominus \mathcal{S}_1$ if the following relation holds for *any* two pairs of (m_1, n_1) and (m_2, n_2):

$$n_1 M - m_1 N = m_2 N - n_2 M, \tag{11}$$

or, equivalently,

$$(n_1 + n_2) M = (m_1 + m_2) N, \tag{12}$$

where $0 \le \{n_1, n_2\} \le N - 1$ and $0 \le \{m_1, m_2\} \le M - 1$. Eq. (11) holds in two cases: (a) $m_1 = m_2 = n_1 = n_2 = 0$. In this case, we have one redundancy between $\mathcal{S}_1 \ominus \mathcal{S}_2$ and $\mathcal{S}_2 \ominus \mathcal{S}_1$ at lag 0. (b) $n_1 + n_2 = kN$ and $m_1 + m_2 = kM$ where k is a positive integer. This condition is satisfied when (n_1, n_2) equals to any of the $N - 1$ pairs of values given as $(1, N-1), (2, N-2), \ldots, (N-1, 1)$ and $(m_1 + m_2)$ equals to any of the $M - 1$ pairs of values $(1, M-1), (2, M-2), \ldots, (M-1, 1)$. Therefore, there are $(N-1)(M-1)+1 = MN - M - N + 2$ redundant pairs between sets $S_1 \ominus S_2$ and $S_2 \ominus S_1$. □

Remarks: From the above two Propositions, we observe:

(a) Because each cross-lag pair has MN unique lags, and there are $MN - M - N + 2$ redundant entries between them, the prototype coprime array has $MN + MN - (MN - M - N + 2) = MN + M + N - 2$ unique lags;

(b) Among these redundant entries, the lag-0 one cannot be avoided. Therefore, for a modified coprime array, the maximum possible number of unique lags is $MN + MN - 1 = 2MN - 1$.

3.2. Proposed Coprime Array with No Cross-lag Redundancies

The modified coprime array design, as shown in Fig. 2, is implemented by changing the inter-element spacing such that the conditions of cross-lag redundancies, described in (11) and (12), are violated. The sensor locations of the two subarrays in the proposed array configuration is expressed as:

$$\begin{aligned} \tilde{\mathcal{S}}_1 &= \{n\tilde{M} \mid 0 \le n \le N - 1\}, \\ \tilde{\mathcal{S}}_2 &= \{m\tilde{N} \mid 0 \le n \le M - 1\}, \end{aligned} \tag{13}$$

where $\tilde{M}$ and $\tilde{N}$ are integers with $\tilde{M} \ge M$ and $\tilde{N} \ge N$.

In the proposed coprime array design, at least one of the conditions $\tilde{N} \ge 2N - 1$ and $\tilde{M} \ge 2M - 1$ is set to be true. Here, $\tilde{M}$ and $\tilde{N}$ are coprime integers whereas the coprimarity between M and N is no longer required. The locations of sensors in the proposed array structure can be expressed as:

$$\begin{aligned} \tilde{\mathcal{S}} &= \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2, \\ &= \{n\tilde{M} \mid 0 \le n \le N - 1\} \cup \{m\tilde{N} \mid 0 \le m \le M - 1\}. \end{aligned} \tag{14}$$

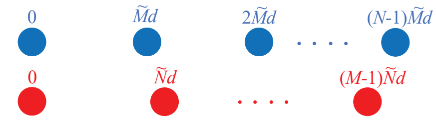


Fig. 2. Proposed array configuration.

Note that the total number of elements in set $\tilde{\mathcal{S}}$ remains $\tilde{Q} = M + N - 1$. The difference co-array of the proposed array having location set $\tilde{\mathcal{S}}$ can be given as:

$$\tilde{\mathcal{D}} = \tilde{\mathcal{S}} \ominus \tilde{\mathcal{S}} = (\tilde{\mathcal{S}}_1 \ominus \tilde{\mathcal{S}}_1) \cup (\tilde{\mathcal{S}}_1 \ominus \tilde{\mathcal{S}}_2) \cup (\tilde{\mathcal{S}}_2 \ominus \tilde{\mathcal{S}}_1) \cup (\tilde{\mathcal{S}}_2 \ominus \tilde{\mathcal{S}}_2), \tag{15}$$

where

$$\begin{aligned} \tilde{S}_1 \ominus \tilde{S}_1 &= \{(n_1 - n_2)\tilde{M}d\} = \{l_1 \tilde{M}d\}, \\ \tilde{S}_1 \ominus \tilde{S}_2 &= \{(n\tilde{M} - m\tilde{N})d\}, \\ \tilde{S}_2 \ominus \tilde{S}_1 &= \{(m\tilde{N} - n\tilde{M})d\}, \\ \tilde{S}_2 \ominus \tilde{S}_2 &= \{(m_1 - m_2)\tilde{N}d\} = \{l_2 \tilde{N}d\}, \end{aligned} \tag{16}$$

where $0 \le \{n, n_1, n_2\} \le N-1$, $0 \le \{m, m_1, m_2\} \le M-1$, $-N+1 \le l_1 \le N-1$, and $-M+1 \le l_2 \le M-1$.

In the sequel, we show through three Propositions that the proposed coprime array design completely eliminates cross-lag redundancies and, as such, achieves $2MN - 1$ unique lags.

Proposition 3: All the self-lags are present in the cross-lag sets, i.e., $((\tilde{S}_1 \ominus \tilde{S}_1) \cup (\tilde{S}_2 \ominus \tilde{S}_2)) \subset ((\tilde{S}_1 \ominus \tilde{S}_2) \cup (\tilde{S}_2 \ominus \tilde{S}_1))$.

Proof. This is similar to Proposition 1 in [6] proved for a generalized oprime array structure, i.e., coprime array with compressed inter-element spacing (CACIS). □

Proposition 4: Each of the cross-lag sets $\tilde{S}_1 \ominus \tilde{S}_2$ and $\tilde{S}_2 \ominus \tilde{S}_1$ contains MN unique elements.

Proof. This can be proved by following the same procedure as in Proposition 1. □

Proposition 5: There is only one cross-lag redundancy in the difference co-array of the proposed array design which exists at the lag position 0.

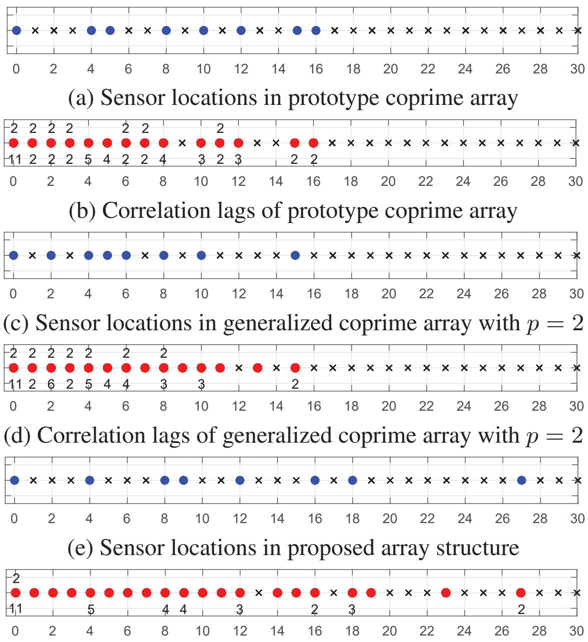
Proof. From Eq. (16), we know that elements in set $\tilde{\mathcal{S}}_1 \ominus \tilde{\mathcal{S}}_2$ overlap with those in $\tilde{\mathcal{S}}_2 \ominus \tilde{\mathcal{S}}_1$ when the following relation is true for any two pairs (m_1, n_1) and (m_2, n_2):

$$n_1 \tilde{M} - m_1 \tilde{N} = m_2 \tilde{N} - n_2 \tilde{M}, \tag{17}$$

or, equivalently,

$$(n_1 + n_2)\tilde{M} = (m_1 + m_2)\tilde{N}, \tag{18}$$

where $0 \le \{n_1, n_2\} \le N - 1, 0 \le \{m_1, m_2\} \le M - 1$. Eqs. (17) and (18) hold in the following two cases: (a) $m_1 = m_2 = n_1 = n_2 = 0$. This corresponds to the redundancy we cannot avoid. (b) $n_1 + n_2 = k\tilde{N}$ and $m_1 + m_2 = k\tilde{M}$ where k is a positive integer. At least one of these conditions



(a) Sensor locations in prototype coprime array

(b) Correlation lags of prototype coprime array

(c) Sensor locations in generalized coprime array with $p = 2$

(d) Correlation lags of generalized coprime array with $p = 2$

(e) Sensor locations in proposed array structure

(f) Correlation lags of the proposed array structure

Fig. 3. Comparison of the difference co-arrays achieved by different sparse array configurations for $M = 4$ and $N = 5$ ($\times$ denotes a hole at the corresponding location).

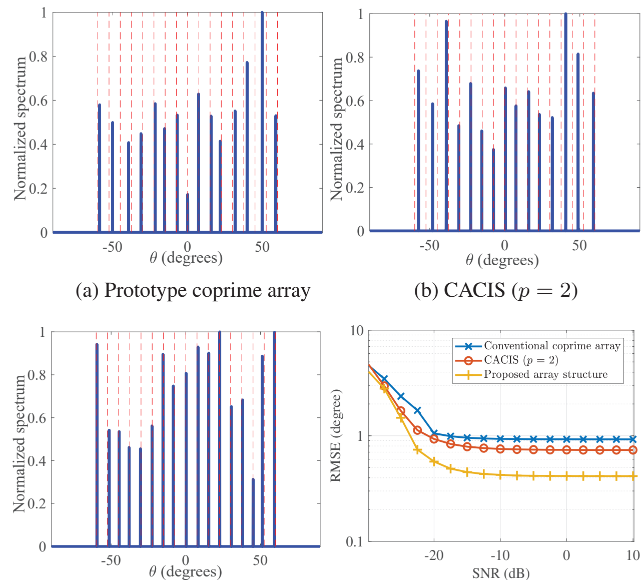
will be violated in the proposed coarray design because either $\tilde{N} \geq 2N - 1$ or $\tilde{M} \geq 2M - 1$ is satisfied in the proposed array design and, as such, either $0 \leq m_1 + m_2 \leq 2M - 2 < \tilde{M}$ or $0 \leq n \leq 2N - 2 < \tilde{N}$ will hold. As a result, there is only one cross-lag redundancy between $\tilde{\mathcal{S}}_1 \ominus \tilde{\mathcal{S}}_2$ and $\tilde{\mathcal{S}}_2 \ominus \tilde{\mathcal{S}}_1$ corresponding to lag 0. $\square$

From the above three Propositions, it is clear that the proposed array design offers $2MN - 1$ unique lags. This is the highest number to be provided in the coprime array framework.

4. SIMULATION RESULTS

In the first simulation, we consider the prototype coprime array, the generalized coprime array (CACIS with $p = 2$) [6], and the proposed array structure consisting of 8 sensors. For all the arrays, we take $M = 4$ and $N = 5$, whereas for the proposed array we select $\tilde{M} = M = 4$ and $\tilde{N} = 2N - 1 = 9$ such that $\tilde{M}$ and $\tilde{N}$ are coprime.

Fig. 3 shows the positions of the physical sensors and the achieved co-array lags for these arrays designs. In Figs. 3(b), 3(d), and 3(f), the numbers shown below the co-array positions denote the number of redundancies in the co-array, whereas the numbers shown above the co-array denote the number of cross-lag redundancies. The numbers of co-array DOFs for these three array configurations are 14, 14, and 20, respectively. It is observed from Fig. 3 that the only cross-lag redundancy present in the proposed array structure exists at lag 0, whereas all other array structures exhibit significant



(a) Prototype coprime array

(b) CACIS ($p = 2$)

(c) Proposed array structure

(d) RMSE performance

Fig. 4. DOA estimation performance of the sparse arrays under consideration.

cross-lag redundancies at several lag positions. Moreover, note that the proposed sparse array has sensors relatively more sparsely spaced; however, the co-array lags are less sparsely spaced compared to co-arrays of the existing sparse array designs.

Figs. 4(a)–4(c) compare the estimated spectra of 17 sources uniformly distributed between -60^{o} 60^{o} for these array configurations, where the LASSO method [28] is used, and all the available cross-lags are utilized. The input signal-to-noise ratio (SNR) is 0 dB and 500 snapshots are used. It is clear that the first two array configurations do not provide correct DOA estimation for all signal arrivals because of lack of DOFs, whereas the proposed coprime array configuration provides sufficient number of DOFs to yield accurate DOA estimates.

In order to compare the root mean squared error (RMSE) performance, we reduce the number of sources to 11 so that all the three array configurations have enough DOFs to perform DOA estimation. It is observed in Fig. 4(d) that the proposed array structure yields the lowest RMSE compared to the other two array configurations.

5. CONCLUSION

We have proposed a novel coprime array design which provides increased number of DOFs compared to the prototype coprime array by modifying the inter-element spacing of the constituting uniform linear subarrays. It is analytically verified that the proposed coprime array design completely eliminates non-zero cross-lag redundancies. Simulation results show that the proposed design provides better DOA estimation performance compared to the existing sparse array designs.

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