Non-Redundant Sparse Array with Flexible Aperture

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***Abstract*—In this paper, we present a novel non-redundant sparse array design that simultaneously achieves the highest possible number of degrees-of-freedom and a flexible array aperture, resulting in superior direction-of-arrival (DOA) estimation performance. We first develop a zero-redundancy design rule based on the co-array properties of the sparse arrays. This design rule guides the construction of non-redundant sparse arrays with a compact as well as a flexible aperture. As a result, given the same number of physical sensors, the proposed array design provides a difference co-array with the maximum number of correlation lags and resolves more sources than the minimum redundancy array. Moreover, the proposed array design provides better DOA estimation resolution compared to the minimum hole array due to the extended array aperture. Simulation results demonstrate the superiority of the proposed array design compared to the existing array structures.**

Keywords: Sparse array, minimum redundancy array, non-redundant array, difference co-array, direction-of-arrival estimation.

I. Introduction

Direction-of-Arrival (DOA) estimation is an important research problem in array signal processing which finds broad applications in radar, sonar, wireless communications, radio astronomy, and many other fields [1, 2]. Due to the Nyquist sampling theorem, uniform linear array (ULA) has traditionally emerged as the commonly used sensor array structure for DOA estimation and its DOA estimation performance has been well analyzed [1–4]. However, ULAs cannot resolve more sources than the number of elements in the sensor array. Therefore, several research efforts have been invested in the past to detect more sources than the number of sensors using sparse arrays by exploiting their difference co-arrays [5–7].

The minimum redundancy array (MRA) and the minimum hole array (MHA) are popular classical sparse array structures. The MRA achieves the maximum number of consecutive difference co-array lags which can be achieved by a sparse array for the given number of sensors [5], whereas the MHA, also known as non-redundant array or Golomb array, minimizes the number of holes in the difference co-array while achieving the maximum number of co-array lags [6, 8]. In other words, the MHA achieves the smallest possible array aperture for the given number of sensors such that the maximum number of co-array lags are achieved. Such properties are attractive to maintain a low sidelobe level when performing DOA estimation for a high number of sources.

Recently, significant research efforts have been dedicated to develop systematical sparse arrays designs which follow a specific design formulation or structure, thus enabling convenient design and analysis. In this context, two notable sparse arrays are the nested array [9] and the coprime array [10]. These array structures and their variants have been extensively analyzed, and closed-form expressions for their design process and the achievable number of degrees-of-freedom (DOFs) are well devised [9–15]. Structured sparse array design and analysis exploiting higher-order statistics [16–18] and frequency diversity [20–23] have also attracted significant attention.

Effective design of non-redundant sparse arrays has remained a topic of great interest due to their ability to provide the highest number of DOFs [6, 8]. Non-redundant sparse arrays generally yield holes in the rendered difference co-array and thus become difficult to exploit all co-array lags for subspace-based DOA estimation methods [3, 4] which, in the context of co-array-based DOA estimation, requires the lags to be consecutive [24]. In contrary, recently developed compressive sensing-based DOA estimation methods can effectively use all the co-array lags [11, 15, 25]. In addition, exploiting Toeplitz structure-based covariance matrix interpolation strategies [26–29] can further provide higher estimation accuracy.

In this paper, we propose a novel non-redundant sparse array design which simultaneously achieves the highest possible number of DOFs and a desired array aperture for DOA estimation. For this purpose, we first develop a zero-redundancy sparse array design rule based on the co-array properties of the sensor arrays. Exploiting this design rule, we synthesize non-redundant sparse arrays which enjoy a desired array aperture that is larger than that of an MHA. Since the DOA estimation resolution depends on the array aperture, the proposed array design achieves enhanced resolution while maintaining the highest possible DOFs for the given number of sensors.

The rest of the paper is organized as follows. Signal model and necessary preliminaries are discussed in Section II. In Section III, we present the zero-redundancy design rule for designing non-redundant sparse arrays. Subsequently, Section IV presents the design approaches of non-redundant array with a flexible aperture. Simulation results are provided in Section V to demonstrate the superiority of the proposed array design compared to MRA and MHA. Finally, conclusions are drawn in Section VI.

Notations: We use lower-case (upper-case) bold characters to denote vectors (matrices). In particular, $\mathbf{I}_N$ denotes the $N \times N$ identity matrix. $(.)^{\mathrm{T}}$ and $(.)^{\mathrm{H}}$ respectively represent the transpose and conjugate transpose of a matrix or a vector. The operator $\mathrm{vec}(\cdot)$ performs vectorization by stacking all columns of a matrix on top of the another, and $\mathrm{diag}(\mathbf{x})$ denotes a diagonal matrix that uses the elements of vector $\mathbf{x}$ as its diagonal elements. In addition, $|\cdot|_F$ and $|\cdot|_1$ denote the Frobenius norm and l_1-norm, respectively. Moreover, $\mathbb{E}[\cdot]$ is the statistical expectation operator and $\otimes$ denotes the Kronecker product.

II. Preliminaries

A. Signal Model

Consider a sparse sensor array consisting of N elements such that the corresponding sensor positions are given by $p_1 \cdot \lambda/2, \cdots,\ p_N \cdot \lambda/2$, where $p_1, \cdots,\ p_N$ represent unique non-negative integers and λ is the signal wavelength. Without loss of generality, we consider the first sensor position as the reference, i.e., $p_1 = 0$. All sensor positions are sorted in an ascending order such that $p_u < p_{u+1}$ for $u = 1, \cdots, N-1$.

Consider Q uncorrelated far-field narrowband signals impinging on the sensor array from distinct angles $\{\theta_1, \cdots, \theta_Q\}$. The baseband signal vector $\mathbf{x}(t)$ received at the sparse sensor array can be expressed as:

$$\mathbf{x}(t) = \sum_{q=1}^{Q} s_q(t)\mathbf{a}(\theta_q) + \mathbf{n}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{s}(t) = [s_1(t), \cdots, s_Q(t)]^{\mathrm{T}}$ with $s_q(t)$ representing the baseband signal of the qth source. Elements in the noise vector $\mathbf{n}(t)$ are considered to be circularly symmetric independent and identically distributed complex white Gaussian random processes which are uncorrelated from the impinging signals. The matrix $\mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_Q)]$ denotes the array manifold where $\mathbf{a}(\theta_q)$ represents the array steering vector in the direction of angle θ_q, given by:

$$\mathbf{a}(\theta_q) = [1, e^{\mathrm{j}\pi p_2 \sin(\theta_q)}, \cdots, e^{\mathrm{j}\pi p_N \sin(\theta_q)}]^{\mathrm{T}}. \tag{2}$$

The covariance matrix $\mathbf{R}_{\mathbf{x}}$ of the received baseband signals $\mathbf{x}(t)$ is obtained as:

$$\begin{aligned}\mathbf{R}_{\mathbf{x}} &= \mathbb{E}[\mathbf{x}(t)\mathbf{x}^{\mathrm{H}}(t)] = \mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^{\mathrm{H}} + \bar{\sigma}_{\mathrm{n}}^2\mathbf{I}_N \\ &= \sum_{q=1}^{Q} \sigma_q^2 \mathbf{a}(\theta_q)\mathbf{a}^{\mathrm{H}}(\theta_q) + \bar{\sigma}_{\mathrm{n}}^2\mathbf{I}_N.\end{aligned} \tag{3}$$

Here, $\bar{\sigma}_{\mathrm{n}}^2$ denotes the noise power and $\mathbf{R}_s = \mathbb{E}[\mathbf{s}(t)\mathbf{s}^{\mathrm{H}}(t)] = \mathrm{diag}([\sigma_1^2, \sigma_2^2, \cdots, \sigma_Q^2])$ is the source covariance matrix with σ_q^2 denoting the power of the qth source. In practice, the covariance matrix is estimated from the sample average of M available samples as:

$$\hat{\mathbf{R}}_{\mathbf{x}} = \frac{1}{M}\sum_{t=0}^{M-1} \mathbf{x}(t)\mathbf{x}^{\mathrm{H}}(t). \tag{4}$$

B. Difference Co-array

Vectorizing the correlation matrix $\mathbf{R}_{\mathbf{x}}$ yields [9, 11]:

$$\mathbf{z} = \mathrm{vec}(\mathbf{R}_{\mathbf{x}}) = \tilde{\mathbf{A}}\mathbf{b} + \bar{\sigma}_{\mathrm{n}}^2\tilde{\mathbf{i}}, \tag{5}$$

where $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}(\theta_1), \ldots, \tilde{\mathbf{a}}(\theta_Q)]$ is the virtual array manifold with $\tilde{\mathbf{a}}(\theta_q) = \mathbf{a}^*(\theta_q) \otimes \mathbf{a}(\theta_q)$ representing the virtual array steering vector. Moreover, $\mathbf{b} = [\sigma_1^2, \sigma_2^2, \cdots, \sigma_Q^2]^{\mathrm{T}}$ is the vector of signal powers and $\tilde{\mathbf{i}} = \mathrm{vec}(\mathbf{I}_N)$. Comparing Eqs. (1) and (5), the new vector $\mathbf{z}$ serves as a single-snapshot received data vector corresponding to a source signal vector $\mathbf{b}$, whereas the noise contribution is represented by a deterministic term, $\bar{\sigma}_{\mathrm{n}}^2\tilde{\mathbf{i}}$.

Denote $\mathbb{P} = \{p_1, \cdots, p_N\}$ as an integer set representing the sensor positions of the sparse array on a half-wavelength grid. The corresponding set representing the difference co-array of $\mathbb{P}$ contains all difference lags that form the virtual array and can be expressed as the following set [11]:

$$\mathbb{D} = \mathbb{P} \ominus \mathbb{P} = \bigcup_{\forall p_l, p_k \in \mathbb{P}} \{p_l - p_k\}, \tag{6}$$

where $\ominus$ represents the difference co-array operator.

If the difference co-array of the sparse sensor array provides η unique co-array lags, the resulting number of achieved DOFs is determined by $(\eta + 1)/2$. The DOFs of the sparse arrays is directly associated with the maximum number of sources that can be successfully resolved. Therefore, it is highly desirable to design sparse sensor arrays which provide a high number of co-array lags [11].

Non-redundant sparse arrays [6] are highly desirable because they provide the highest number of co-array lags for a given number of sensors. All the non-zero co-array lags produced by the non-redundant sparse arrays are unique. Regardless of the sensor positions, any array consisting of N sensors yields N entries of lag-0 self-lags which cannot be avoided. Therefore, for a non-redundant sparse array consisting of N physical sensors, we can achieve a total number of N^2 co-array lags among which N lags are positioned at 0. Therefore, an N-sensor non-redundant sparse array achieves $N^2 - N + 1$ unique co-array lags.

III. Zero-Redundancy Design Rule

In this section, we present the zero-redundancy design rule which serves as a guideline for designing non-redundant sparse arrays. This design rule is used in the next section to extract non-redundant sparse arrays with a flexible aperture. The motivation behind the flexible array aperture is that the large aperture enables DOA estimation with a higher resolution compared to the conventional MHA that assumes a smallest array aperture for a given number of sensors.

Let us express the sensor positions in a non-redundant sparse array in terms of half wavelength as a vector $\mathbf{p} = [p_1, p_2, \cdots, p_N]^{\mathrm{T}}$. In order for a sparse array to be non-redundant, the position vector $\mathbf{p}$ needs to render the maximum possible number of unique lags. This condition is achieved only if the difference co-array of the designed sensor array contains no lag redundancies except at lag 0.

We observe in Eq. (6) that redundancies exist in the resulting co-array if different pairs of sensor positions produce the same co-array lags. Ignoring the redundancies at lag 0, all co-array lags are unique only if the co-array lag generated from a pair of sensor elements is not equal to the co-array lag generated by another pair of sensor elements. This can be mathematically expressed as:

$$\begin{aligned} p_i - p_j \neq p_k - p_l, \quad & i, j, k, l = 1, \cdots, N, \\ & i \neq j, k \neq l, j \neq l. \end{aligned} \tag{7}$$

Note that the conditions $i \neq j$ and $k \neq l$ ensure that the co-array lags at position 0 are ignored, whereas the condition $j \neq l$ ensures that the condition (7) is checked only for different pair of sensor elements. Since sensor positions $p_1, \cdots, p_N$ are sorted in an ascending order, the condition (7) can be modified to obtain the following zero-redundancy design rule:

$$p_u + 1 \leq p_{u+1} \text{ and } p_i - p_j \neq p_k - p_l, \quad i \neq j, k \neq l, j \neq l, \tag{8}$$

where $i, j, k, l = 1, \cdots, N$ and $u = 1, \cdots, N-1$. Ensuring the zero-redundancy condition in Eq. (8) will result in a non-redundant sparse array.

IV. Non-Redundant Sparse Array with Flexible Aperture

In this section, we exploit the zero-redundancy design rule to design non-redundant sparse arrays with a flexible aperture. First, we show the applicability of the zero-redundancy design rule in constructing an MHA. Subsequently, the design procedure is modified to obtain non-redundant sparse arrays with a flexible array aperture. Since the resolution of a sensor array depends on the array aperture, such a flexible design can significantly improve the DOA estimation performance of the resulting array by enabling an extended array aperture while ensuring that the maximum possible number of co-array lags are achieved.

A. Design of Minimum Hole Array

First, we design MHA by constructing a non-redundant sparse array which achieves the minimum array aperture for the given number of sensors. Exploiting the design rule in Eq. (8), we can formulate the following optimization problem to achieve this objective:

$$\begin{array}{lll} \min\limits_{p_n \in \mathbb{Z}^+, n \in \{2,\cdots,N\}} & p_N & \\ \text{subject to} & p_u + 1 \leq p_{u+1}, & u = 1, \cdots, N-1, \\ & p_i - p_j \neq p_k - p_l, & i \neq j, k \neq l, j \neq l, \\ & & i, j, k, l = 1, \cdots, N. \end{array} \tag{9}$$

Here, $\mathbb{Z}^+$ represents the set of positive integers. The above optimization problem ensures a minimum array aperture and achieves the maximum possible number of co-array lags which can be produced by the given number of sensors.

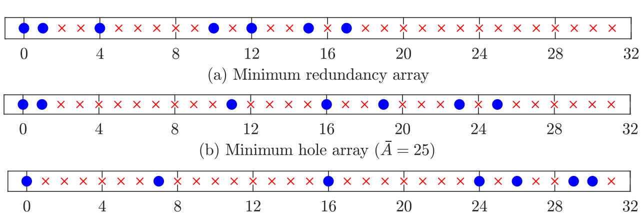
Note that the difference co-arrays are symmetric in nature. Therefore, the zero-redundancy design rule only needs to be employed for either the positive or negative co-array lags. This can be achieved by modifying the optimization problem (9) for the positive side of co-array as follows:

$$\begin{array}{lll} \min\limits_{p_n \in \mathbb{Z}^+, n \in \{2,\cdots,N\}} & p_N & \\ \text{subject to} & p_u + 1 \leq p_{u+1}, & u = 1, \cdots, N-1, \\ & p_i - p_j \neq p_k - p_l, & i > j, k > l, j \neq l, \\ & & i, k = 2, \cdots, N, \\ & & j, l = 1, \cdots, N-1. \end{array} \tag{10}$$

The conditions $i > j$ and $k > l$ in the above optimization ensure that lags $p_i - p_j$ and $p_k - p_l$ are guaranteed to be positive. Thus, the optimization problem (10) only checks the zero-redundancy design rule for positive lags which automatically ensures the uniqueness of negative lags due to the co-array symmetry. The optimization problems (9) and (10) result in the same MHA.

B. Non-Redundant Array with Desired Array Aperture

We now generalize the concept of non-redundant sparse array design to further enjoy the important feature of flexible apertures. Consider the desired array aperture for a non-redundant sparse array to be at least A_{flex} such that $\bar{A} < A_{\text{flex}}$, where $\bar{A}$ is the array aperture of the corresponding MHA.



(a) Minimum redundancy array

(b) Minimum hole array ($\bar{A} = 25$)

(c) Proposed non-redundant sparse array ($A_{\text{flex}} = 30$)

Fig. 1. Sensor positions for sparse arrays being compared ($N = 7$ sensors).

We modify the optimization problem (10) to design a non-redundant sparse array with aperture A_{flex} as follows:

$$\begin{array}{lll} \min\limits_{p_n \in \mathbb{Z}^+, n \in \{2,\cdots,N\}} & p_N & \\ \text{subject to} & p_N \geq A_{\text{flex}} & \\ & p_u + 1 \leq p_{u+1}, & u = 1, \cdots, N-1, \\ & p_i - p_j \neq p_k - p_l, & i > j, k > l, j \neq l, \\ & & i, k = 2, \cdots, N, \\ & & j, l = 1, \cdots, N-1. \end{array} \tag{11}$$

The above optimization yields a non-redundant sparse array which has an aperture of at least A_{flex}. Any suitable non-linear optimization methods like integer-based genetic algorithm [30] or exhaustive search can be used to solve these optimizations. Furthermore, references [6, 8] discuss useful strategies to reduce the computational cost of such exhaustive search strategies.

For DOA estimation, we utilize Eq. (5) along with LASSO [31], resulting in the following constrained l_1-norm minimization:

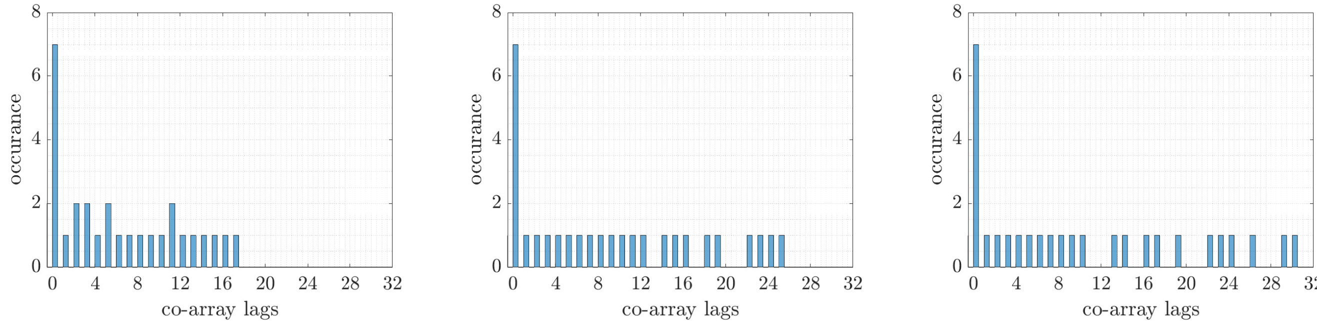
$$\hat{\mathbf{r}} = \arg\min_{\mathbf{r}} \ |\mathbf{Br} - \mathbf{b}|_F^2 + \eta |\mathbf{r}|_1, \tag{12}$$

where $\mathbf{B}$ is an overcomplete dictionary matrix consisting of a grid of steering vectors given by $[\tilde{\mathbf{a}}(\theta_1), \cdots, \tilde{\mathbf{a}}(\theta_G)]$ corresponding to angles $\theta_1, \cdots, \theta_G$ with $G \gg Q$. Moreover, $\mathbf{r}$ is a sparse vector which represents weights to select and add the desired steering columns from $\mathbf{B}$ to reconstruct the single-snapshot vector $\mathbf{b}$. Furthermore, η is the regularization parameter which trades off between the Frobenius norm-based fitting and the l_1-norm-based sparsity measure.

V. Numerical Results

In this section, we compare the performance of the proposed non-redundant sparse array having a flexible aperture with the classical MRA and MHA. As an example, we consider sparse arrays consisting of $N = 7$ physical sensors and the sensor positions for the three array configurations under consideration are plotted in Fig. 1. The non-negative co-array weights for these arrays are also illustrated in Fig. 2. It can be observed that the MRA achieves consecutive continuous lags. However, it yields 4 co-array redundancies in the positive axis of difference co-array. On the other hand, all the positive co-array lags provided by the MHA are unique and the resulting array achieves an aperture of 25. In comparison, the proposed non-redundant sparse array with a flexible aperture not only provides unique positive co-array lags but also achieves a desired large aperture, which is considered to be $A_{\text{flex}} = 30$ in this case. While achieving the maximum number of DOFs, the proposed array design enables high resolution DOA estimation due to its large aperture compared to the classical MRA and MHA.

In order to compare the DOA estimation performance in terms of resolving closely spaced sources, we first consider $Q = 7$ sources which are uniformly distributed between $-9°$ and $9°$. The input signal-to-noise ratio (SNR) is fixed at 0 dB and 500 data snapshots are used. Fig. 3 shows the LASSO spectra for the sparse arrays under consideration. It is evident that both MRA and MHA fail to resolve all the closely spaced sources. On the other hand, the proposed non-redundant sparse array with a flexible aperture of $A_{\text{flex}} = 30$ is able to resolve



(a) Minimum redundancy array (b) Minimum hole array (c) Proposed non-redundant array

Fig. 2. Co-array weight functions of the sparse arrays under consideration ($N = 7$ sensor, $\bar{A} = 25$, $A_{\text{flex}} = 30$).

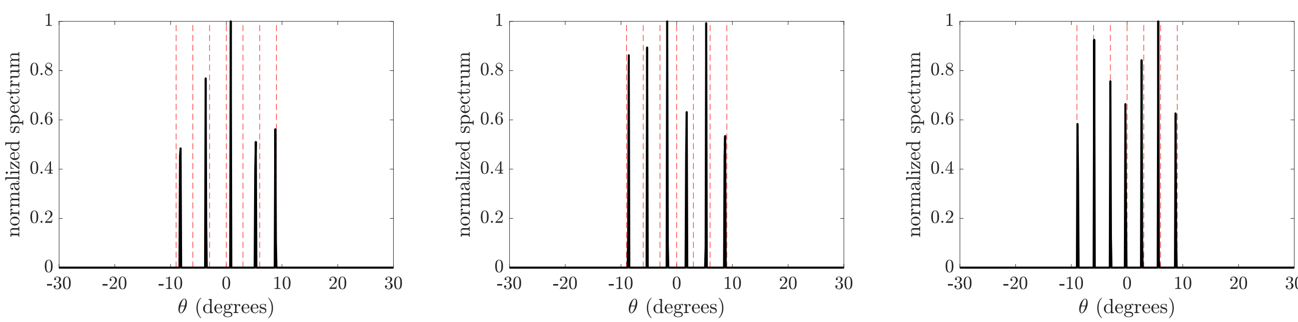
all the sources successfully, thereby verifying its capability to significantly improve the DOA estimation.

In the second simulation example, we increase the number of sources to $Q = 16$, and the sources are uniformly spaced between $-30°$ and $30°$. All other parameters are kept the same. It is observed in Fig. 4 that both MRA and MHA fail to successfully resolve all the incoming sources. However, the proposed non-redundant sparse array with a large array aperture of $A_{\text{flex}} = 30$ successfully resolves all the sources and provides fine DOA estimation results.

The above simulation results evidently confirm that the non-redundant sparse array design with flexible aperture achieves superior DOA resolution characteristics due to the large array aperture and yields enhanced DOA estimation performance.

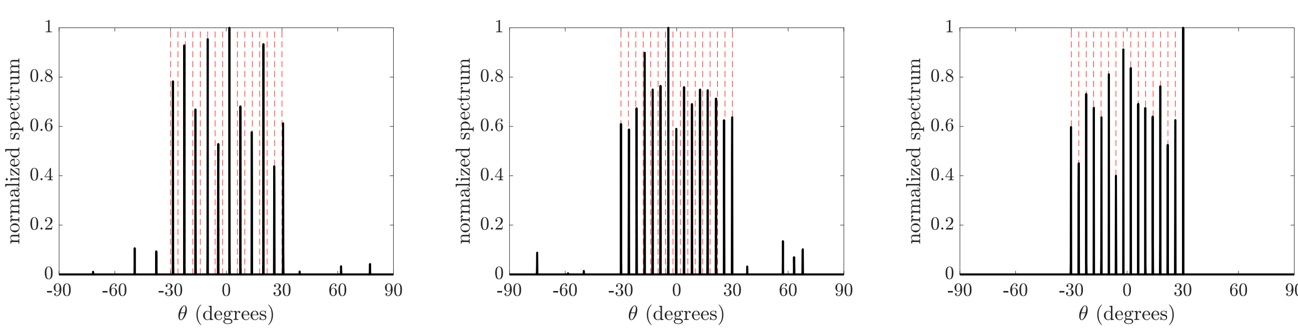
VI. Conclusions

In this paper, we presented non-redundant sparse arrays that provide the highest possible number of DOFs for a given number of physical sensors and achieve the desired array aperture. A zero-redundancy design rule is devised and is used to guide the design of non-redundant sparse arrays with flexible aperture. The rendered array designs offer high resolution DOA estimation and outperform conventional MRA and MHA configurations.



(a) Minimum redundancy array (b) Minimum hole array (c) Proposed non-redundant array

Fig. 3. LASSO spectra for $Q = 7$ sources uniformly distributed from $-9°$ to $9°$ using the sparse arrays consisting of $N = 7$ sensors ($M = 500$ snapshots, SNR = 0 dB, $\bar{A} = 25$, $A_{\text{flex}} = 30$).



(a) Minimum redundancy array (b) Minimum hole array (c) Proposed non-redundant array

Fig. 4. LASSO spectra for $Q = 16$ sources uniformly distributed from $-30°$ to $30°$ using the sparse arrays consisting of $N = 7$ sensors ($M = 500$ snapshots, SNR = 0 dB, $\bar{A} = 25$, $A_{\text{flex}} = 30$).

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